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Topological Yang–Mills Theory with Two Fermionic Charges. A Superfield Approach on Kähler Manifolds

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Abstract

The four-dimensional topological Yang–Mills theory with two anticommuting charges is naturally formulated on Kähler manifolds. By using a superspace approach we clarify the structure of the Faddeev–Popov sector and determine the total action. This enables us to perform perturbation theory around any given instanton configuration by manifestly maintaining all the symmetries of the topological theory. The superspace formulation is very useful for recognizing a trivial observable (i.e. having vanishing correlation functions only) as the highest component of a gauge invariant superfield. As an example of non-trivial observables we construct the complete solution to the simultaneous cohomology problem of both fermionic charges. We also show how this solution has to be used in order to make Donaldson's interpretation possible.

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1 Introduction

There is a renewed interest in topological Yang–Mills (TYM) theory [1] over the past two years. While an immediate physical meaning is still a matter of debate (see however [2], [3]), TYM certainly offers powerful methods [4], [5] for extracting non-perturbative information [6], [7] about supersymmetric chromodynamics.

Before presenting the content of our paper we would like to give a short review of TYM. For more details one should consult the excellent articles [8], [9].

1.1 Review of Topological Yang–Mills Theory

The basic property of TYM is that its action can be written as the variation of some gauge invariant expression. The variation itself acts on the gauge field as a shift and is nilpotent. The former property allows one to formulate TYM on curved manifolds.

There are essentially two ways of deriving the TYM action, each of them having its own merit. One can, for instance, construct an action incorporating the following set of subsidiary conditions: self-duality (instanton), fixing of the topological shift and the BRS gauge fixing [10], [11]. Being directly related to instanton calculus it can be conveniently used for explicit calculations [12], [13].

Another way [1] to obtain TYM is by twisting the Euclidean $N = 2$ supersymmetric gauge theory and by coupling thereafter to external gravity. While the last step breaks down the original $N = 2$ supersymmetry, some supersymmetries may exist on the curved background as global symmetries. We call them fermionic symmetries.

A single fermionic symmetry is preserved on arbitrary Riemannian manifolds. One can show [14] that two or four symmetries remain unbroken if the manifold is Kähler or hyper-Kähler, respectively, because each Kähler structure is equivalent to a corresponding Killing spinor.

For the rest of this short review the TYM has a single fermionic symmetry q, hence it is formulated on a Riemannian four-manifold. The action produced by twisting differs by q-exact terms from that obtained via subsidiary conditions. However, the correlation functions are the same [12].

The TYM obtained by twisting gains in clarity when formulated in superspace [15] despite the consequence that BRS symmetry has to be introduced in terms of superfields.

TYM imposes very strong restrictions on the possible observables. Only those objects which are not highest components of gauge invariant superfields can have non-trivial correlations. The proper mathematical background for constructing such observables is equivariant cohomology [16]. Its use in TYM has been initiated by [17] and further developped in [18], [19].

The only known examples of TYM observables are the Donaldson polynomials [20], (for a review see [21]). They can be obtained [22], [23] by twisting some components of the $N = 2$ superconformal anomaly [24]. By an appropriate change of renormalization prescriptions one can obtain the Donaldson polynomials in the semiclassical approximation. However, this approximation turns out to be sufficient as a consequence of the non-renormalization theorem for chiral fields [25], [26].

Further restrictions on the observables come through the path integral representation. For TYM there is a canonical functional measure [1] $[\mathrm{d}m][\mathrm{d}\widetilde{m}]$, where m denotes the moduli and $\widetilde{m}$ their q-transformation. The above measure is supplemented with simple prescriptions for integrating out all non-zero modes [1] (as well as some zero modes [13]). The integration over the fermionic zero modes amounts to replacing $\widetilde{m}$ everywhere by $\mathrm{d}m$, such that the correlation function becomes the integral of a certain top-form over the moduli space [1].

Another consequence of the fermionic zero modes $\widetilde{m}$ is the vanishing of the partition function. This means that the action of TYM has an Abelian global invariance—hereafter called ghost number symmetry—that is not shared by the path integral. The amount ΔU of violation of the ghost number symmetry is obtained by integrating the twisted $\mathbb{R}$-anomaly [23], [27] over the manifold. One can choose the ghost number U such that its variation ΔU exactly compensates the dimension of the moduli space of self-dual instantons [28]. This provides us with the following selection rule: The total ghost number of the observables in a non-vanishing correlation function equals the dimension of the instanton moduli space.

1.2 Content of the Paper

In the present paper we consider TYM with two anticommutig symmetries along with the items described previously. The twisting appropriate for this case has been found for the first time in [14]. Due to the existence of a Killing spinor which is necessary to preserve the second supersymmetry the Riemannian manifold has an integrable complex structure whose Kähler form is closed, thus being in fact

a Kähler manifold.

We use this formulation in section 2 to set up a superspace approach. By extending the method of [15] to two Grassmann variables we encounter constraints in superspace. They are solved in terms of a prepotential, a pair of chiral-antichiral connections and a chiral-antichiral pair of antisymmetric tensors. As a consequence, the gauge symmetry is replaced by the local chiral symmetry. Furthermore, the gauge symmetry is completely fixed if one is postulating subsidiary conditions for the chiral connections. The gauge and Faddeev–Popov terms are then easily constructed.

Section 3 deals with the symmetries of the theory. On Kähler manifolds each conservation law involves two contragredient (with respect to conservation indices) but inequivalent tensors. They belong to different superfields and have different positions within each multiplet. Most prominent examples are the energy-momentum tensor and the pair of fermionic symmetry currents. These objects represent (up to improvement terms which leave the conservation law unchanged) the highest components of gauge invariant superfields.

The 'classical' theory under discussion is invariant under two global Abelian symmetries, whereas one encounters anomalies at the quantum level. In section 4 we use this opportunity to check our superspace formalism in perturbation theory. We evaluate the gravitational contribution to both Abelian anomalies and found that they are equal. Moreover, they are equal to the dimension of the instanton moduli space. The result coincides with the one obtained for a Riemannian manifold admitting a Kähler structure.

In section 5 we give the solution to the simultaneous cohomology problem of both fermionic charges. In contrast to TYM with only one fermionic symmetry, this solution consists of local observables which may depend on a larger number of fields, including antisymmetric fields. We can, however, show that the correlation functions of the integrated observables can be interpreted as integrals of top-forms over the moduli space of selfdual instantons.

A different treatment of TYM with two fermionic charges has been given in [29] and [30]. A short comparision with our work is provided at the end of section 5.

2 Superspace Approach

Consider a superspace with two Grassmann coordinates θ, $\bar{\theta}$ related by complex conjugation $\mathrm{i}\bar{\theta} = \overline{\mathrm{i}\theta}$. The commuting coordinates z^m and $z^{\bar{m}}$ with $m = 1, 2$, $\bar{m} = \bar{1}, \bar{2}$ represent local holomorphic and antiholomorphic coordinates, respectively on a Kähler manifold $\mathcal{K}$ without boundary and endowed with the metric $(1,1)$-form

$$\gamma = g_{m\bar{n}} \mathrm{d}z^m \mathrm{d}z^{\bar{n}} = \partial\bar{\partial} h \; . \tag{2.1}$$

Here, h is the Kähler potential of the metric.

To these coordinates one associates the superconnections A_m, $A_{\bar{m}} = -(A_m)^\dagger$, A_θ and $A_{\bar{\theta}} = -(A_\theta)^\dagger$, in short A_M with $M = m, \bar{m}, \theta, \bar{\theta}$. Under an infinitesimal gauge transformation they change as

$$\delta A_M = \mathrm{D}_M K = \nabla_M + [A_M, K] \tag{2.2}$$

where $K = -K^\dagger$. We use ∇_m, $\nabla_{\bar{m}}$ to represent Kähler derivatives, while θ, $\bar{\theta}$ directions are flat, i.e.$\nabla_\theta = \partial_\theta$, $\nabla_{\bar{\theta}} = \partial_{\bar{\theta}}$.

We also introduce the following covariant superfields: the anti-Hermitean scalar $\Lambda = -\Lambda^\dagger$ and the pair of complex conjugate, antisymmetric and anticommuting tensors X_{mn}, $X_{\bar{m}\bar{n}} = -(X_{mn})^\dagger$. They transform as

$$\delta X = [K, X] \tag{2.3}$$

where $X = \Lambda, X_{mn}$ or $X_{\bar{m}\bar{n}}$.

All superfields are taken in an irreducible representation of the compact and semi-simple group G with anti-Hermitean generators $t_a = -t_a^\dagger$ where $a = 1, \ldots, n = \dim \mathrm{G}$ and with totally antisymmetric structure constants c_{abc} defined through $[t_a, t_b] = c_{abc} t_c$. The generators t_a are normalized by $\mathrm{Tr}(t_a t_b) = -\delta_{ab}$.

From superconnections one can construct the superfield strengths F_{mn}, $F_{m\theta}$, $F_{m\bar{\theta}}$, $F_{\theta\theta}$, their complex conjugates, $F_{m\bar{n}}$ and $F_{\theta\bar{\theta}}$. They are defined by

$$F_{MN} = \nabla_M A_N + A_M A_N - (-)^{|M||N|} (M \leftrightarrow N) \tag{2.4}$$

where $|M|$ is the grading of M, i.e. $|M| = 0$ for $M = m, \bar{m}$ and $|M| = 1$ for $M = \theta, \bar{\theta}$. The superfield strengths transform covariantly, i.e. as (2.3).

2.1 Constraints in Superspace

In order to obtain the desired TYM one imposes the constraints

$$F_{\theta\theta} = F_{\bar\theta\bar\theta} = F_{m\bar\theta} = F_{\bar m\theta} = 0\ ; \tag{2.5}$$

$$\mathrm{D}_{\bar\theta} X_{mn} + iF_{mn} = 0\ ; \qquad \mathrm{D}_\theta X_{\bar m\bar n} + iF_{\bar m\bar n} = 0\ . \tag{2.6}$$

Notice the similarity of (2.5) with the constraints in superspace for the $N = 1$ supersymmetric gauge multiplet [31]. The role of (2.5) is to ensure that TYM has a single gauge field.

The fermionic symmetries are represented by the Grassmann derivatives $\mathrm{q} = \partial_\theta$ and $\bar{\mathrm{q}} = \partial_{\bar\theta}$. They are nilpotent and anticommute: $\mathrm{q}^2 = \bar{\mathrm{q}}^2 = \{\mathrm{q}, \bar{\mathrm{q}}\} = 0$.

2.2 Wess–Zumino Gauge

We define the fields of TYM through covariant superfields and their Grassmann covariant derivatives:

$$\begin{array}{rclrcl}
\psi_m &=& -\,F_{m\theta}|\ ; & \psi_{\bar m} &=& -\,F_{\bar m\bar\theta}|\ ; \\
\varphi &=& -\mathrm{i}\ F_{\theta\bar\theta}|\ ; & \lambda &=& \Lambda|\ ; \\
g_+ &=& 2\ \mathrm{D}_\theta\Lambda|\ ; & g_- &=& 2\ \mathrm{D}_{\bar\theta}\Lambda|\ ; \\
k &=& [\mathrm{D}_\theta, \mathrm{D}_{\bar\theta}]\ \Lambda|\ ; & f_{m\bar n} &=& F_{m\bar n}|\ ; \\
\chi_{mn} &=& X_{mn}|\ ; & \chi_{\bar m\bar n} &=& X_{\bar m\bar n}|\ ; \\
b_{mn} &=& \mathrm{i}\mathrm{D}_\theta X_{mn}|\ ; & b_{\bar m\bar n} &=& \mathrm{i}\mathrm{D}_{\bar\theta} X_{\bar m\bar n}|\ ; \\
f_{mn} &=& \mathrm{i}\mathrm{D}_{\bar\theta} X_{mn}|\ ; & f_{\bar m\bar n} &=& \mathrm{i}\mathrm{D}_\theta X_{\bar m\bar n}|\ .
\end{array} \tag{2.7}$$

The vertical bar means the lowest component of the superfield, i.e. at $\theta = \bar\theta = 0$.

Further components of these superfields can be obtained from (2.7) with the help of Bianchi identities and constraints (2.5), (2.6). They are related to (2.7) through the gauge covariant derivatives D_m, $\mathrm{D}_{\bar m}$ or vanish.

The transformations of the fields defined by (2.7) are generated by the Grassmann covariant derivatives

$$\delta\,(X|) = \mathrm{i}\,(\zeta + \mathrm{i}\xi)\ \mathrm{D}_\theta X| + \mathrm{i}\,(\zeta - \mathrm{i}\xi)\ \mathrm{D}_{\bar\theta} X|\ . \tag{2.8}$$

Here X represents any of the covariant superfields on the rhs. of (2.7); ζ and ξ are real, anticommuting parameters.

From (2.8) one can see that the fermionic transformation $\mathrm{q} = \partial_\theta$ (or $\bar{\mathrm{q}} = \partial_{\bar\theta}$ resp.) is always accompanied by a gauge transformation $[A_\theta, X]|$ (or $[A_{\bar\theta}, X]|$ resp.) restoring the gauge covariance. From the transformation of the various field strength f_{mn}, $f_{\bar m\bar n}$, and $f_{m\bar n}$ one can deduce how the gauge fields are transforming, albeit up to a gauge transformation. Usually, in the Wess–Zumino (WZ) gauge the last gauge transformation is suppressed. A simple calculation leads to the following transformation rules:

$$
\begin{aligned}
\delta a_m &= \mathrm{i}(\zeta+\mathrm{i}\xi)\,\psi_m\;; & \delta a_{\bar m} &= \mathrm{i}(\zeta-\mathrm{i}\xi)\,\psi_{\bar m}\;;\\
\delta\psi_m &= (\zeta-\mathrm{i}\xi)\,\mathrm{D}_m\varphi\;; & \delta\psi_{\bar m} &= (\zeta+\mathrm{i}\xi)\,\mathrm{D}_{\bar m}\varphi\;;\\
\delta\varphi &= 0\;; & &\\
\delta\lambda &= \tfrac{\mathrm{i}}{2}(\zeta+\mathrm{i}\xi)\,g_+ + \tfrac{\mathrm{i}}{2}(\zeta-\mathrm{i}\xi)\,g_-; & &\\
\delta g_+ &= -\mathrm{i}(\zeta-\mathrm{i}\xi)\,(k-\mathrm{i}[\varphi,\lambda])\;; & \delta g_- &= -\mathrm{i}(\zeta+\mathrm{i}\xi)\,(k+\mathrm{i}[\varphi,\lambda]);\\
\delta k &= \tfrac{1}{2}(\zeta+\mathrm{i}\xi)\,[\varphi,g_+] - \tfrac{1}{2}(\zeta-\mathrm{i}\xi)\,[\varphi,g_-]\;; & &\\
\delta\chi_{mn} &= (\zeta+\mathrm{i}\xi)\,b_{mn} + (\zeta-\mathrm{i}\xi)\,f_{mn}\;; & &\\
\delta\chi_{\bar m\bar n} &= (\zeta+\mathrm{i}\xi)\,f_{\bar m\bar n} + (\zeta-\mathrm{i}\xi)\,b_{\bar m\bar n}\;; & &\\
\delta b_{mn} &= -\mathrm{i}(\zeta-\mathrm{i}\xi)\,(\mathrm{D}_m\psi_n - \mathrm{D}_n\psi_m + [\varphi,\chi_{mn}])\;; & &\\
\delta b_{\bar m\bar n} &= -\mathrm{i}(\zeta+\mathrm{i}\xi)\,(\mathrm{D}_{\bar m}\psi_{\bar n} - \mathrm{D}_{\bar n}\psi_{\bar m} + [\varphi,\chi_{\bar m\bar n}])\;. & &
\end{aligned}
\tag{2.9}
$$

This choice of the field components enables one to declare them (Kähler) metric independent. It follows that fermionic symmetries and the variation with respect to the metric always commute.

The transformations (2.9) close into φ-field dependent gauge transformations.

In section 5 we shall give the fermionic transformations generated by the nilpotent, anticommutative operations q and $\bar{\mathrm{q}}$.

2.3 Action

The action in superspace is given by

$$
\mathcal{S} = \tfrac{1}{4}\int_{\mathcal{K}} \mathrm{d}^2 z\,\mathrm{d}^2\bar z\; g\;\partial_\theta\partial_{\bar\theta}\;\mathrm{Tr}\left\{-\tfrac{1}{4}X_{mn}X^{mn} + \Lambda\,(\mathrm{i}F - [\mathrm{D}_\theta,\mathrm{D}_{\bar\theta}]\Lambda)\right\} \tag{2.10}
$$

where $g = \det g_{m\bar n}$ and $F = g^{\bar n m}F_{m\bar n}$. Indices are raised by the inverse Kähler metric $g^{\bar n m}$ of the Kähler manifold $\mathcal{K}$.

The equations of motion in superspace are

$$
\begin{array}{rcll}
\mathrm{D}_\theta X_{mn} & = & 0\ ; & \mathrm{D}_{\bar\theta} X_{\bar m\bar n} = 0\ ; \\
\frac{1}{2}\,\mathrm{D}^n X_{mn} + \mathrm{D}_\theta \mathrm{D}_m \Lambda & = & 0\ ; & \frac{1}{2}\,\mathrm{D}^{\bar n} X_{\bar m\bar n} + \mathrm{D}_\theta \mathrm{D}_{\bar m} \Lambda = 0\ ; \\
F + 2\mathrm{i}[\mathrm{D}_\theta, \mathrm{D}_{\bar\theta}]\Lambda & = & 0\ ; & \\
\{\mathrm{D}_m, \mathrm{D}^m\}\Lambda + \frac{\mathrm{i}}{2}\{X_{mn}, X^{mn}\} & + & \multicolumn{2}{l}{4\mathrm{i}\{\mathrm{D}_\theta\Lambda, \mathrm{D}_{\bar\theta}\Lambda\} - 2\mathrm{i}\,[\Lambda, [\Lambda, F_{\theta\bar\theta}]] = 0\ .}
\end{array}
\tag{2.11}
$$

From (2.10) one can get the action in component fields:

$$
\begin{aligned}
\mathcal{S} \;=\; & \tfrac{1}{8}\int_{\mathcal{K}} \mathrm{d}^2 z \mathrm{d}^2 \bar z\ g\ \mathrm{Tr}\Big\{\lambda\{\mathrm{D}_m, \mathrm{D}^m\}\varphi + \mathrm{i} f k - k^2 + \tfrac{1}{2}\,(f_{mn} f^{mn} \\
& - b_{mn} b^{mn}) + \mathrm{i}\,(g_+ \mathrm{D}_m \psi^m + g_- \mathrm{D}^m \psi_m - \chi_{mn} \mathrm{D}^m \psi^n - \chi^{mn} \mathrm{D}_m \psi_n) \\
& + 2\mathrm{i}\lambda\{\psi_m, \psi^m\} + \varphi\left(\tfrac{1}{2}\{\chi_{mn}, \chi^{mn}\} + \{g_+, g_-\}\right) - [\varphi, \lambda]^2\Big\}\ .
\end{aligned}
\tag{2.12}
$$

It coincides (up to some field redefinition) with that obtained in [14] by twisting $N=2$ supersymmetric Yang–Mills theory (SYM).

2.4 Solution of the Constraints

One cannot solve the constraints (2.6) in superspace. Instead one can take them into account by means of Lagrange multipliers. For this purpose one introduces a new action

$$
\begin{aligned}
\widetilde{\mathcal{S}} \;=\; & \mathcal{S} + \tfrac{1}{16}\int_{\mathcal{K}} \mathrm{d}^2 z \mathrm{d}^2 \bar z\, g\ \left(\partial_\theta \mathrm{Tr}\left\{L^{mn}\left(\mathrm{D}_{\bar\theta} X_{mn} + \mathrm{i} F_{mn}\right)\right\}\right. \\
& \left. + \,\partial_{\bar\theta} \mathrm{Tr}\left\{L_{mn}\left(\mathrm{D}_\theta X^{mn} + \mathrm{i} F^{mn}\right)\right\}\right)
\end{aligned}
\tag{2.13}
$$

where L_{mn} and $L_{\bar m\bar n} = -(L_{mn})^\dagger$ are a pair of complex conjugate, anticommuting, and antisymmetric superfields satisfying

$$
\mathrm{D}_\theta L_{mn} = \mathrm{D}_{\bar\theta} L_{\bar m\bar n} = 0\ . \tag{2.14}
$$

The (covariant) superfields entering (2.13) are subject to the constraints (2.5) and (2.14).

The solution of these constraints can be given in terms of

- a Hermitean prepotential $V = V^\dagger$,

- a chiral-antichiral pair of superconnections ϕ_m, $\phi_{\bar m} = -(\phi_m)^\dagger$ depending on a single Grassmann variable $\partial_{\bar\theta}\phi_m = \partial_\theta\phi_{\bar m} = 0$, and
- a pair of chiral-antichiral, anticommuting, and antisymmetric superfields M_{mn}, $M_{\bar m\bar n} = -(M_{mn})^\dagger$ obeying $\partial_\theta M_{mn} = \partial_{\bar\theta} M_{\bar m\bar n} = 0$.

It can be presented in the form

$$\begin{array}{rclrcl} A_\theta &=& \mathrm{e}^{-\frac{V}{2}}\partial_\theta \mathrm{e}^{\frac{V}{2}}\ ; & A_{\bar\theta} &=& \mathrm{e}^{\frac{V}{2}}\partial_{\bar\theta}\mathrm{e}^{-\frac{V}{2}}\ ; \\ A_m &=& \mathrm{e}^{\frac{V}{2}}\left(\phi_m + \nabla_m\right)\mathrm{e}^{-\frac{V}{2}}\ ; & A_{\bar m} &=& \mathrm{e}^{-\frac{V}{2}}\left(\phi_{\bar m} + \nabla_{\bar m}\right)\mathrm{e}^{\frac{V}{2}}\ ; \\ L_{mn} &=& \mathrm{e}^{-\frac{V}{2}} M_{mn}\mathrm{e}^{\frac{V}{2}}\ ; & L_{\bar m\bar n} &=& \mathrm{e}^{\frac{V}{2}} M_{\bar m\bar n}\mathrm{e}^{-\frac{V}{2}}\ . \end{array} \tag{2.15}$$

The constraint superfields V, ϕ_m, $\phi_{\bar m}$, M_{mn}, and $M_{\bar m\bar n}$ are determined up to local chiral transformations.

2.5 BRS Symmetry in Superspace

It is convenient to describe the chiral transformations by a BRS (nilpotent) operation. For this purpose one introduces a pair of chiral-antichiral, anticommuting superfields C, $C^\dagger$ with $\partial_{\bar\theta} C = \partial_\theta C^\dagger = 0$. The unconstrained prepotentials transform as follows:

$$\begin{array}{rclrcl} \mathrm{s}\mathrm{e}^V &=& \mathrm{e}^V C + C^\dagger \mathrm{e}^V\ ; & & & \\ \mathrm{s}\phi_m &=& \mathcal{D}_m C\ ; & \mathrm{s}\phi_{\bar m} &=& -\mathcal{D}_{\bar m} C^\dagger\ ; \\ \mathrm{s}M_{mn} &=& \{C^\dagger, M_{mn}\}\ ; & \mathrm{s}M_{\bar m\bar n} &=& -\{C, M_{\bar m\bar n}\} \end{array} \tag{2.16}$$

where $\mathcal{D}_m$ (or $\mathcal{D}_{\bar m}$ resp.) is the gauge covariant derivative constructed with ϕ_m (or $\phi_{\bar m}$ resp.). The gauge symmetry is represented by an anti-Hermitean Faddeev–Popov ghost superfield

$$K = \frac{C - C^\dagger}{2} + \tanh \pounds_{\frac{V}{4}} \left(\frac{C + C^\dagger}{2}\right) \tag{2.17}$$

where $\pounds_X = [X,\ ]$ denotes the Lie bracket of the superfield X. The 'matter' transforms as

$$\begin{array}{c} \mathrm{s}\Lambda = -[K, \Lambda]\ ; \\ \mathrm{s}X_{mn} = -\{K, X_{mn}\}\ ; \qquad \mathrm{s}X_{\bar m\bar n} = -\{K, X_{\bar m\bar n}\}\ . \end{array} \tag{2.18}$$

The local chiral symmetry is fixed if only the corresponding connections ϕ_m, $\phi_{\bar{m}}$ obey subsidiary conditions, e.g. $\nabla^m \phi_m = \nabla_m \phi^m = 0$. In order to find the gauge fixing and Faddeev–Popov-terms one introduces a pair of chiral-antichiral (commuting) superfields D, $D^\dagger$ with $\partial_{\bar{\theta}} D = \partial_\theta D^\dagger = 0$, as well as their BRS variations $B = \mathrm{s}D$ and $B^\dagger = \mathrm{s}D^\dagger$. The superfields B, $B^\dagger$ are anticommuting and form a chiral-antichiral pair. They serve as Lagrange multipliers for the gauge fixing conditions characterized by the real parameter α. The BRS (trivial) term of the action is

$$
\begin{aligned}
\mathcal{S}' &= -\tfrac{1}{4}\,\mathrm{s}\int_{\mathcal{K}} \mathrm{d}^2 z \mathrm{d}^2 \bar{z}\, g \left(\partial_\theta \mathrm{Tr} \left\{ D \left(\nabla^m \phi_m - \alpha \partial_\theta B \right) \right\} \right. \\
& \qquad \left. + \partial_{\bar{\theta}} \mathrm{Tr} \left\{ D^\dagger \left(\nabla_m \phi^m - \alpha \partial_\theta B^\dagger \right) \right\} \right) .
\end{aligned}
\tag{2.19}
$$

The total action is $\widetilde{S} + S'$ and represents the starting point of all perturbative or non-perturbative considerations.

2.6 Correlation Functions

We would like to study correlation functions of the form

$$
\left\langle \prod_i \mathcal{O}_i \right\rangle = \int [\mathrm{d}\tilde{\mu}] \prod_i \mathcal{O}_i \, \exp\left\{ -\frac{1}{e^2} \left(\widetilde{S} + S' \right) \right\} \tag{2.20}
$$

where $\mathcal{O}_i$ are gauge invariant ($\mathrm{s}\mathcal{O}_i = 0$) and metric independent polynomials in the fields, $[\mathrm{d}\tilde{\mu}]$ denotes the path integral measure of all unconstrained superfields, i.e. V, ϕ_m, $\phi_{\bar{m}}$, M_{mn}, $M_{\bar{m}\bar{n}}$, X_{mn}, $X_{\bar{m}\bar{n}}$, Λ, C, $C^\dagger$, D, $D^\dagger$, B, and $B^\dagger$; e is the gauge coupling constant.

If one assumes that $\mathrm{q}\mathcal{O}_i = \bar{\mathrm{q}}\mathcal{O}_i = 0$, the correlation functions are independent of the coupling constant. The reason for this property is the form of the total action

$$
\widetilde{S} + S' = \mathrm{q}\bar{\mathrm{q}}\mathcal{V} + \mathrm{q}\mathcal{W} - \bar{\mathrm{q}}\bar{\mathcal{W}} \tag{2.21}
$$

where $\mathcal{V}$ is a Hermitean superfield and $\mathcal{W}$, $\bar{\mathcal{W}}$ is a pair of complex conjugate chiral-antichiral anticommuting superfields, i.e. $\bar{\mathrm{q}}\mathcal{W} = \mathrm{q}\bar{\mathcal{W}} = 0$.

Many observables are constructed from the (total) action by operations which commute with both q and $\bar{\mathrm{q}}$ and therefore have the form (2.21). They are highest components of BRS invariant superfields. Of course, they are q and $\bar{\mathrm{q}}$ invariant. Observables which are highest components can be shown to have vanishing correlation functions only.

3 Symmetries

Most of the symmetries of supersymmetric field theories are encoded in the supercurrent [32], a multiplet containing the energy-momentum tensor, the supersymmetry and the $\mathbb{R}$-symmetry currents. For $N = 2$ supersymmetry an additional isovector current [24] corresponding to the automorphism $SU(2)$ symmetry group belongs to the supermultiplet. (Of course, the number of supersymmetry currents is doubled.)

The twisting procedure enables one to get the above currents for the TYM and to recast the result into appropriate superfields [23], [33]. The method has the advantage to be easily applicable [34] to any $N = 2$ supersymmetric gauge theory, e.g. to super-Yang–Mills coupled to relaxed hypermultiplet [35].

The approach we shall use below is adapted to the relative simple structure of the TYM obtained from pure $N = 2$ supersymmetric Yang–Mills theory.

3.1 Energy-Momentum Tensor

Consider a variation of the metric in the action (2.12). The canonical energy-momentum tensor defined by

$$\delta_g \mathcal{S} = -\tfrac{1}{8} \int_{\mathcal{K}} \mathrm{d}^2 z \mathrm{d}^2 \bar{z}\, g\, \delta g^{\bar{n}m} \vartheta_{m\bar{n}} \tag{3.1}$$

has the form

$$\begin{aligned} \vartheta_{m\bar{n}} \;=\; & \mathrm{Tr}\{-\mathrm{i}k f_{m\bar{n}} + \mathrm{D}_m\lambda \mathrm{D}_{\bar{n}}\varphi + \mathrm{D}_m\varphi \mathrm{D}_{\bar{n}}\lambda - \mathrm{i}\,(\psi_m \mathrm{D}_{\bar{n}} g_- + \psi_{\bar{n}} \mathrm{D}_m g_+) \\ & -2\mathrm{i}\lambda\{\psi_m, \psi_{\bar{n}}\} - g_{m\bar{n}}[k^2 - \mathrm{i}kf + \mathrm{D}_p\lambda \mathrm{D}^p\varphi + \mathrm{D}_p\varphi \mathrm{D}^p\lambda \\ & -\mathrm{i}\,(\psi_p \mathrm{D}^p g_- + \psi^p \mathrm{D}_p g_+) - 2\mathrm{i}\lambda\{\psi_p, \psi^p\} - \mathrm{i}\varphi\{g_+, g_-\} \\ & +[\varphi, \lambda]^2]\}\ . \end{aligned} \tag{3.2}$$

It is the last component of the superfield $-2\mathrm{i}\mathrm{Tr}\,\{\Lambda\,[F_{m\bar{n}} - g_{m\bar{n}}\,(F + \mathrm{i}[\mathrm{D}_\theta, \mathrm{D}_{\bar{\theta}}]\Lambda)]\}$. The on-shell version of the latter reads

$$Q_{m\bar{n}} = -2\mathrm{i}\mathrm{Tr}\left\{\Lambda\left(F_{m\bar{n}} - \tfrac{1}{2} g_{m\bar{n}} F\right)\right\} \tag{3.3}$$

and obeys the conservation laws

$$\begin{aligned} \partial_\theta \nabla^{\bar{n}} Q_{m\bar{n}} + \nabla^n J_{mn} &= 0\ ; \\ \partial_{\bar{\theta}} \nabla^n Q_{n\bar{m}} + \nabla^{\bar{n}} J_{\bar{m}\bar{n}} &= 0 \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} J_{mn} &= \mathrm{iTr}\left\{\tfrac{1}{2}F_{(m}{}^{p}X_{n)p} + 2F_{m\theta}\mathrm{D}_n\Lambda - \Lambda\mathrm{D}_{[m}F_{n]\theta}\right\} \; ; \\ J_{\bar{m}\bar{n}} &= \mathrm{iTr}\left\{\tfrac{1}{2}F^{\bar{p}}{}_{(\bar{m}}X_{\bar{n})\bar{p}} - 2F_{\bar{m}\bar{\theta}}\mathrm{D}_{\bar{n}}\Lambda + \Lambda\mathrm{D}_{[\bar{m}}F_{\bar{n}]\bar{\theta}}\right\} \; . \end{aligned} \tag{3.5}$$

In deriving these expressions we used the equations of motion (2.11); hence J_{mn} and its complex conjugate $J_{\bar{m}\bar{n}}$ have no definite off-shell continuation like $Q_{m\bar{n}}$.

The energy-momentum tensor $\vartheta_{m\bar{n}}$ obeys the conservation laws

$$\nabla^n\vartheta_{mn} + \nabla^{\bar{n}}\vartheta_{m\bar{n}} = 0 \; ; \qquad \nabla^n\vartheta_{n\bar{m}} + \nabla^{\bar{n}}\vartheta_{\bar{m}\bar{n}} = 0 \tag{3.6}$$

where

$$\begin{aligned} \vartheta_{mn} &= \mathrm{Tr}\left\{-\tfrac{1}{2}f_{(m}{}^{p}f_{n)p} + \mathrm{D}_{(m}\lambda\mathrm{D}_{n)}\varphi + \mathrm{i}\chi_{np}\mathrm{D}_m\psi^p - \mathrm{i}\psi_n\mathrm{D}_m g_-\right\} \; ; \\ \vartheta_{\bar{m}\bar{n}} &= \mathrm{Tr}\left\{\tfrac{1}{2}f^{\bar{p}}{}_{(\bar{m}}f_{\bar{n})\bar{p}} + \mathrm{D}_{(\bar{m}}\lambda\mathrm{D}_{\bar{n})}\varphi + \mathrm{i}\chi_{\bar{n}\bar{p}}\mathrm{D}_{\bar{m}}\psi^{\bar{p}} - \mathrm{i}\psi_{\bar{n}}\mathrm{D}_{\bar{m}}g_+\right\} \; . \end{aligned} \tag{3.7}$$

The conservation partners ϑ_{mn}, $\vartheta_{\bar{m}\bar{n}}$ of the energy-momentum tensor are higher components of the superfields J_{mn} and $J_{\bar{m}\bar{n}}$ but obviously not highest component like $\vartheta_{m\bar{n}}$.

The correlation functions of $\vartheta_{m\bar{n}}$ are vanishing. From the special structure of (3.4) it follows that the correlation functions of all the components of $\nabla^n J_{mn}$, $\nabla^{\bar{n}} J_{\bar{m}\bar{n}}$ (and in particular of $\nabla^n\vartheta_{mn}$, $\nabla^{\bar{n}}\vartheta_{\bar{m}\bar{n}}$) vanish.

Of course, radiative corrections may alter some of the above conclusions. For Riemannian manifolds one can however show that the energy-momentum tensor remains highest component. In particular, there is no contribution to the energy-momentum trace coming from the (Riemannian) manifold [36]. We expect a similar property for TYM on Kähler manifolds.

3.2 Fermionic Symmetries

Consider the fermionic transformations (2.9) with $z, \bar{z}$-dependent parameters ζ, ξ and pick up terms proportional to (Kähler) derivatives of these parameters. The fermionic symmetry currents defined by

$$\begin{aligned} \delta\mathcal{S} = \; & \tfrac{\mathrm{i}}{16}\int_{\mathcal{K}} \mathrm{d}^2z\mathrm{d}^2\bar{z}\, g\left[\nabla^m(\zeta + \mathrm{i}\xi)\, s_m^{-\mathrm{can}} + \nabla^{\bar{m}}(\zeta + \mathrm{i}\xi)\, s_{\bar{m}}^{-\mathrm{can}}\right. \\ & \left. + \nabla^m(\zeta - \mathrm{i}\xi)\, s_m^{+\mathrm{can}} + \nabla^{\bar{m}}(\zeta - \mathrm{i}\xi)\, s_{\bar{m}}^{+\mathrm{can}}\right] \end{aligned} \tag{3.8}$$

are

$$
\begin{aligned}
s_m^{+\mathrm{can}} &= 2\mathrm{Tr}\left\{f_{mn}\psi^n - g_-\mathrm{D}_m\varphi\right\} \ ; \\
s_{\bar m}^{+\mathrm{can}} &= 2\mathrm{Tr}\left\{\chi_{\bar m\bar n}\mathrm{D}^{\bar n}\varphi + \mathrm{i}k\psi_{\bar m} + \varphi[\lambda,\psi_{\bar m}]\right\} \ ; \\
s_m^{-\mathrm{can}} &= 2\mathrm{Tr}\left\{\chi_{mn}\mathrm{D}^{n}\varphi - \mathrm{i}k\psi_{m} + \varphi[\lambda,\psi_{m}]\right\} \ ; \\
s_{\bar m}^{-\mathrm{can}} &= 2\mathrm{Tr}\left\{f_{\bar m\bar n}\psi^{\bar n} - g_+\mathrm{D}_{\bar m}\varphi\right\} \ .
\end{aligned}
\tag{3.9}
$$

Due to the equations of motion (2.11) they obey the conservation rules

$$
\nabla^m s_m^{\pm\mathrm{can}} + \nabla^{\bar m} s_{\bar m}^{\pm\mathrm{can}} = 0 \ . \tag{3.10}
$$

In order to find the superfields whose components are related to fermionic symmetry currents we investigate at first the global symmetries of the action.

3.3 Global Symmetries

The action (2.10) is invariant under a global $SU(2)$

$$
\begin{aligned}
\delta\psi_m &= -v\psi_m - u_{mn}\psi^n \ ; & \delta\psi_{\bar m} &= v\psi_{\bar m} - u_{\bar m\bar n}\psi^{\bar n} \ ; \\
\delta\chi_{mn} &= -v\chi_{mn} - u_{mn}g_- \ ; & \delta\chi_{\bar m\bar n} &= -v\chi_{\bar m\bar n} - u_{\bar m\bar n}g_+ \ ; \\
\delta g_+ &= -vg_+ + \tfrac{1}{2}u_{mn}\chi^{mn} \ ; & \delta g_- &= vg_- + \tfrac{1}{2}u_{\bar m\bar n}\chi^{\bar m\bar n}
\end{aligned}
\tag{3.11}
$$

and under a global Abelian group:

$$
\begin{aligned}
\delta\psi_m &= u\psi_m \ ; & \delta\psi_{\bar m} &= u\psi_{\bar m} \ ; \\
\delta\varphi &= 2u\varphi \ ; & \delta\lambda &= -2u\lambda \ ; \\
\delta\chi_{mn} &= -u\chi_{mn} \ ; & \delta\chi_{\bar m\bar n} &= -u\chi_{\bar m\bar n} \ ; \\
\delta g_+ &= -ug_+ \ ; & \delta g_- &= -ug_- \ .
\end{aligned}
\tag{3.12}
$$

The parameters u, v are real and u_{mn}, $u_{\bar m\bar n}$ are antisymmetric and complex conjugate to each other, i.e. $u_{\bar m\bar n} = -\overline{u_{mn}}$. The corresponding currents are given by

$$
\begin{aligned}
\delta\mathcal{S} = \ & \tfrac{1}{8}\int_{\mathcal{K}} \mathrm{d}^2z\mathrm{d}^2\bar z\, g\left[\tfrac{1}{8}\left(\nabla^p u^{mn} b_{mnp} + \nabla^{\bar p} u^{mn} b_{mn\bar p}\right.\right. \\
& \left. + \nabla^p u^{\bar m\bar n} b_{\bar m\bar n p} + \nabla^{\bar p} u^{\bar m\bar n} b_{\bar m\bar n\bar p}\right) \\
& \left. -\mathrm{i}\left(\nabla^m u b_m + \nabla^{\bar m} u b_{\bar m}\right) - \left(\nabla^m v b'_m + \nabla^{\bar m} v b'_{\bar m}\right)\right] \ .
\end{aligned}
\tag{3.13}
$$

The conserved combinations $b_m \pm b'_m$ and $b_{\bar m} \pm b'_{\bar m}$ are responsible for two global Abelian symmetry groups $\mathbb{R}_\pm$. Through their charges they associate to each field two additive (real) quantum numbers $r_\pm$.

One can still modify the above combinations of currents without disturbing their conservation. A convenient choice is to improve $b_{\bar m} + b'_{\bar m}$ to a q variation and $b_m - b'_m$ to a $\bar{\mathrm{q}}$ variation. The improved currents, denoted by $b^+_{\bar m}$ and b^-_m are defined by

$$b^+_{\bar m} = -4\mathrm{iq}\mathrm{Tr}\{\lambda\psi_{\bar m}\}\ ; \qquad b^-_m = -4\mathrm{i}\bar{\mathrm{q}}\mathrm{Tr}\{\lambda\psi_m\}\ . \tag{3.14}$$

The last step is to construct the conservation partners b^+_m and $b^-_{\bar m}$. The superfields are then easily guessed to be

$$\begin{aligned} B^+_m &= 4\mathrm{i}\mathrm{Tr}\left\{\tfrac{1}{2}F^n{}_{\bar\theta}X_{mn} - F_{\theta\bar\theta}\mathrm{D}_m\Lambda\right\}\ ; \\ B^+_{\bar m} &= 4\mathrm{i}\partial_\theta\mathrm{Tr}\{\Lambda F_{\bar m\bar\theta}\} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} B^-_m &= 4\mathrm{i}\partial_{\bar\theta}\mathrm{Tr}\{\Lambda F_{m\theta}\}\ ; \\ B^-_{\bar m} &= 4\mathrm{i}\mathrm{Tr}\left\{\tfrac{1}{2}F^{\bar n}{}_{\theta}X_{\bar m\bar n} - F_{\theta\bar\theta}\mathrm{D}_{\bar m}\Lambda\right\}\ . \end{aligned} \tag{3.16}$$

They obey the conservation rule

$$\nabla^m B^\pm_m + \nabla^{\bar m} B^\pm_{\bar m} = 0\ . \tag{3.17}$$

The first components of $B^\pm_m$, $B^\pm_{\bar m}$ are the $\mathbb{R}_\pm$-currents discussed above; none of them is observable since none of them is simultaneously annihilated by both q and $\bar{\mathrm{q}}$. The next components are the improved currents of the fermionic symmetries:

$$\begin{aligned} s^+_m &= 2\mathrm{Tr}\left\{f_{mn}\psi^n + \varphi\mathrm{D}_m g_-\right\}\ ; \\ s^+_{\bar m} &= -\mathrm{Tr}\left\{f\psi_{\bar m} + 2g_-\mathrm{D}_{\bar m}\varphi + 2\varphi[\lambda,\psi_{\bar m}]\right\}\ ; \\ s^-_m &= -\mathrm{Tr}\left\{-f\psi_m + 2g_+\mathrm{D}_m\varphi + 2\varphi[\lambda,\psi_m]\right\}\ ; \\ s^-_{\bar m} &= 2\mathrm{Tr}\left\{f_{\bar m\bar n}\psi^{\bar n} + \varphi\mathrm{D}_{\bar m}g_+\right\}\ . \end{aligned} \tag{3.18}$$

One can easily check that eqs. (3.18) differ from (3.9) by terms which do not violate the conservation law.

Of the list (3.18) only $s^+_{\bar m}$ and s^-_m are local observables since they are both q and $\bar{\mathrm{q}}$ invariant. However, being the highest component of the superfields $4\mathrm{i}\mathrm{Tr}\{\Lambda F_{\bar m\bar\theta}\}$ and $4\mathrm{i}\mathrm{Tr}\{\Lambda F_{m\theta}\}$ respectively, all their correlation functions vanish.

There are two more conserved currents which correspond to commuting parameters u_{mn}, $u_{\bar{m}\bar{n}}$ [23], [37]. They give rise to the antichiral superfields

$$
\begin{aligned}
B_{mnp} &= 4\mathrm{i}\mathrm{Tr}\left\{X_{mn}F_{p\theta}\right\} \ ; \\
B_{mn\bar{p}} &= 8\mathrm{i}g_{[m\bar{p}}\partial_\theta \mathrm{Tr}\left\{\Lambda F_{n]\theta}\right\}
\end{aligned}
\tag{3.19}
$$

and to their complex conjugates (chiral ones)

$$
\begin{aligned}
B_{\bar{m}\bar{n}p} &= 8\mathrm{i}g_{p[\bar{m}}\partial_{\bar{\theta}} \mathrm{Tr}\left\{\Lambda F_{\bar{n}]\bar{\theta}}\right\} \ ; \\
B_{\bar{m}\bar{n}\bar{p}} &= 4\mathrm{i}\mathrm{Tr}\left\{X_{\bar{m}\bar{n}}F_{\bar{p}\bar{\theta}}\right\} \ .
\end{aligned}
\tag{3.20}
$$

The conservation rules read

$$
\begin{aligned}
\nabla^p B_{mnp} + \nabla^{\bar{p}} B_{mn\bar{p}} &= 0 \ ; \\
\nabla^p B_{\bar{m}\bar{n}p} + \nabla^{\bar{p}} B_{\bar{m}\bar{n}\bar{p}} &= 0 \ .
\end{aligned}
\tag{3.21}
$$

Two new local observables emerge:

$$
\begin{aligned}
s_{mnp} &= \partial_{\bar{\theta}} B_{mnp}| &&= 4\mathrm{Tr}\left\{\chi_{mn}\mathrm{D}_p\varphi - f_{mn}\psi_p\right\} \ ; \\
s_{\bar{m}\bar{n}\bar{p}} &= \partial_\theta B_{\bar{m}\bar{n}\bar{p}}| &&= 4\mathrm{Tr}\left\{\chi_{\bar{m}\bar{n}}\mathrm{D}_{\bar{p}}\varphi - f_{\bar{m}\bar{n}}\psi_{\bar{p}}\right\} \ .
\end{aligned}
\tag{3.22}
$$

They are highest components of an antichiral or chiral superfield, respectively, and therefore have vanishing correlation functions.

3.4 Taking BRS into Account

Also if BRS symmetry is taken into account, the trivial observables discussed previously remain higher components of gauge invariant superfields. The starting point is the total action $\tilde{\mathcal{S}}+\mathcal{S}'$, which includes also the Lagrange multipliers L_{mn}, $L_{\bar{m}\bar{n}}$, the Faddeev–Popov ghosts C, $C^\dagger$, and their antighosts D, $D^\dagger$. The dynamics of the total action is somewhat intricate—the superfields X_{mn}, $X_{\bar{m}\bar{n}}$ are set equal to the corresponding Lagrange multipliers

$$
X_{mn} = L_{mn} \ ; \qquad X_{\bar{m}\bar{n}} = L_{\bar{m}\bar{n}} \ , \tag{3.23}
$$

therefore becoming covariantly chiral. Some equations of motion (see (2.11)) have to be modified:

$$\begin{array}{rcl} \mathrm{D}^n L_{mn} + 2\mathrm{D}_\theta \mathrm{D}_m \Lambda + 2\mathrm{i}\mathrm{e}^{-\frac{V}{2}} \left(\nabla_m B^\dagger - \left[C^\dagger, \nabla_m D^\dagger\right]\right) \mathrm{e}^{\frac{V}{2}} & = & 0\ ; \\ \mathrm{D}^{\bar{n}} L_{\bar{m}\bar{n}} + 2\mathrm{D}_{\bar{\theta}} \mathrm{D}_{\bar{m}} \Lambda + 2\mathrm{i}\mathrm{e}^{\frac{V}{2}} \left(\nabla_{\bar{m}} B + [C, \nabla_{\bar{m}} D]\right) \mathrm{e}^{-\frac{V}{2}} & = & 0\ . \end{array} \tag{3.24}$$

The gauge is fixed by

$$\nabla^m \phi_m = 2\alpha \partial_\theta B\ ; \qquad \nabla^{\bar{m}} \phi_{\bar{m}} = 2\alpha \partial_{\bar{\theta}} B^\dagger\ . \tag{3.25}$$

Finally, ghosts and antighosts obey the equations

$$\begin{array}{rclcrcl} \nabla^m \mathcal{D}_m C & = & 0\ ; & \qquad & \nabla^{\bar{m}} \mathcal{D}_{\bar{m}} C^\dagger & = & 0\ ; \\ \mathcal{D}_m \nabla^m D & = & 0\ ; & \qquad & \mathcal{D}_{\bar{m}} \nabla^{\bar{m}} D^\dagger & = & 0\ . \end{array} \tag{3.26}$$

After some calculation one gets the following system of BRS modified current superfields corresponding to the fermionic symmetries:

$$\begin{array}{rcl} \widetilde{B}^+_m & = & 4\mathrm{i}\mathrm{Tr}\left\{\frac{1}{2} F^n{}_{\bar{\theta}} X_{mn} - F_{\theta\bar{\theta}} \mathrm{D}_m \Lambda + \mathrm{is}\left(\nabla_m D^\dagger \mathrm{e}^V \partial_{\bar{\theta}} \mathrm{e}^{-V}\right)\right\}\ ; \\ \widetilde{B}^+_{\bar{m}} & = & 4\mathrm{i}\mathrm{Tr}\left\{\partial_\theta \left(\Lambda F_{\bar{m}\bar{\theta}}\right) - \mathrm{is}\left(D^\dagger \partial_{\bar{\theta}} \phi_{\bar{m}}\right)\right\}\ ; \\ \widetilde{B}^-_m & = & 4\mathrm{i}\mathrm{Tr}\left\{\partial_{\bar{\theta}} \left(\Lambda F_{m\theta}\right) - \mathrm{is}\left(D \partial_\theta \phi_m\right)\right\}\ ; \\ \widetilde{B}^-_{\bar{m}} & = & 4\mathrm{i}\mathrm{Tr}\left\{\frac{1}{2} F^{\bar{n}}{}_\theta X_{\bar{m}\bar{n}} - F_{\theta\bar{\theta}} \mathrm{D}_{\bar{m}} \Lambda + \mathrm{is}\left(\nabla_{\bar{m}} D \mathrm{e}^{-V} \partial_\theta \mathrm{e}^V\right)\right\}\ . \end{array} \tag{3.27}$$

The observable currents are $\widetilde{s}^+_{\bar{m}}$ and $\widetilde{s}^-_m$. The differences $\widetilde{s}^+_{\bar{m}} - s^+_{\bar{m}}$ and $\widetilde{s}^-_m - s^-_m$ are gauge variations of the second component of the chiral superfields $-4\mathrm{Tr}\{D^\dagger \partial_{\bar{\theta}} \phi_{\bar{m}}\}$ and $-4\mathrm{Tr}\{D \partial_\theta \phi_m\}$, respectively.

4 Perturbative Check

One would like to have some confirmation about the correctness of the field theoretic description of TYM on Kähler manifolds. In the present section we shall compute in perturbation theory the gravitational contribution to $\mathbb{R}_\pm$-anomalies. The latter can be considered as finite radiative corrections to the conservation law for the Abelian currents. It will turn out that the anomalies are equal. When integrated over the Kähler manifold they both yield the (formal) dimension of the

instanton moduli space. The resulting number is well known in the mathematical literature and has a standard representation by local polynomials in the curvature.

On the other hand $\mathbb{R}_{\pm}$-anomalies get perturbative contributions from all the (super)fields of the model. Hence, one can in fact check the proposed superspace description.

Since our calculation is limited to one-loop approximation, we shall regularize the $\mathbb{R}_{\pm}$-currents by point-splitting. This way all the symmetries which are supposed to be preserved at the quantum level will be automatically taken into account.

4.1 Green Functions

The Green functions can be obtained from the linearized equations of motion with sources. The source term that is added to the total action has the form

$$\begin{aligned} \tfrac{1}{4}\int_{\mathcal{K}} \mathrm{d}^2z\mathrm{d}^2\bar{z}g\Big\{&\partial_\theta \mathrm{Tr}\left(\tfrac{1}{2}K_{mn}M^{mn} + J^m\phi_m + J_B B + J_C C + J_D D\right) \\ &+\partial_{\bar\theta}\mathrm{Tr}\left(\tfrac{1}{2}K^{mn}M_{mn} + J_m M^m + J_{B^\dagger}B^\dagger + J_{C^\dagger} + J_{D^\dagger}D^\dagger\right) \\ &+\partial_\theta\partial_{\bar\theta}\mathrm{Tr}\left(\tfrac{1}{2}J^{mn}X_{mn} + \tfrac{1}{2}J_{mn}X^{mn} + J_\Lambda\Lambda + JV\right)\Big\} \end{aligned} \tag{4.1}$$

where the sources K_{mn}, $K_{\bar m\bar n}$, J_m, $J_{\bar m}$, J_B, $J_{B^\dagger}$, J_C, $J_{C^\dagger}$, J_D, $J_{D^\dagger}$, J_{mn}, $J_{\bar m\bar n}$, J_Λ, J and the corresponding superfields $M_{\bar m\bar n}$, M_{mn}, $\phi_{\bar m}$, ϕ_m, B, $B^\dagger$, C, $C^\dagger$, D, $D^\dagger$, $X_{\bar m\bar n}$, X_{mn}, Λ, V have the same symmetry and reality (or chirality) properties.

The linearized equations of motion are

$$\begin{aligned} \tfrac{1}{2}\nabla^n M_{mn} + \nabla_m\left(B^\dagger - \mathrm{i}\partial_\theta\Lambda\right) &= J_m\ ; \\ \tfrac{1}{2}\nabla^{\bar n} M_{\bar m\bar n} + \nabla_{\bar m}\left(B - \mathrm{i}\partial_{\bar\theta}\Lambda\right) &= J_{\bar m}\ ; \end{aligned}$$

$$\begin{aligned} \mathrm{i}\nabla_{[m}\phi_{n]} + \partial_{\bar\theta}X_{mn} &= 2K_{mn}\ ; & \mathrm{i}\nabla_{[\bar m}\phi_{\bar n]} + \partial_\theta X_{\bar m\bar n} &= 2K_{\bar m\bar n}\ ; \\ X_{mn} &= M_{mn} + 2J_{mn}\ ; & X_{\bar m\bar n} &= M_{\bar m\bar n} + 2J_{\bar m\bar n}\ ; \\ \nabla^m\phi_m &= 2\alpha\partial_\theta B + J_B\ ; & \nabla_m\phi^m &= 2\alpha\partial_{\bar\theta}B^\dagger + J_{B^\dagger}\ ; \\ \nabla_m\nabla^m C &= J_D\ ; & \nabla^m\nabla_m C^\dagger &= J_{D^\dagger}\ ; \\ \nabla_m\nabla^m D &= J_C\ ; & \nabla^m\nabla_m D^\dagger &= J_{C^\dagger}\ ; \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
\nabla_m \nabla^m V + \nabla_m \phi^m - \nabla^m \phi_m + 4\mathrm{i}\partial_\theta \partial_{\bar\theta} \Lambda + \mathrm{i} J_\Lambda &= 0 \; ; \\
\nabla_m \nabla^m \Lambda + \mathrm{i} J &= 0 \; .
\end{aligned}
$$

Eqs. (4.2) can be converted into differential equations for the Green functions (GF). The solution can be expressed in terms of two basic GF, the scalar GF $G(z,z')$ and the vector one $G_{m\bar n'}(z,z')$ the definition of which is given by

$$
\begin{aligned}
\nabla_p \nabla^p G(z,z') &= -\delta(z,z') \; ; \\
\nabla_p \nabla^p G_{m\bar n'}(z,z') &= -g_{m\bar n'}(z,z')\delta(z,z')
\end{aligned}
\tag{4.3}
$$

where

$$
\delta(z,z') = g^{-1}\delta^2(z-z')\delta^2(\bar z - \bar z') \; . \tag{4.4}
$$

The factor $g_{m\bar n'}(z,z')$ is explained below in section 4.2.

By taking $\mathcal{K}$ a compact Kähler manifold and by observing that the Laplacian $\nabla_p\nabla^p$ is an elliptic operator one can assume that eqs. (4.3) have unique solutions.

We choose to work in the gauge $\alpha = 1$. Below we list only those Green functions which are necessary for the computation of anomalies:

$$
\begin{aligned}
\langle X_{mna} \phi_{\bar p' b} \rangle &= 2\mathrm{i}\delta_{ab}(\bar\theta - \bar\theta')\nabla_{[m} G_{n]\bar p'}(z,z') \; ; \\
\langle X_{\bar m \bar n a} \phi_{p' b} \rangle &= 2\mathrm{i}\delta_{ab}(\theta - \theta')\nabla_{[\bar m} G_{p' \bar n]}(z,z') \; ; \\
\langle \Lambda_a V'_b \rangle &= \mathrm{i}\delta_{ab}(\theta-\theta')(\bar\theta - \bar\theta') G(z,z') \; ; \\
\langle \phi_{ma} B'_b \rangle &= -\delta_{ab}(\theta - \theta')\nabla_m G(z,z') \; ; \\
\left\langle \phi_{\bar m a} \bar B'_b \right\rangle &= -\delta_{ab}(\bar\theta - \bar\theta')\nabla_{\bar m} G(z,z') \; ; \\
\langle V_a B'_b \rangle &= -\delta_{ab}(\theta-\theta') G(z,z') \; ; \\
\left\langle V_a \bar B'_b \right\rangle &= -\delta_{ab}(\bar\theta - \bar\theta') G(z,z') \; ; \\
\langle C_a D'_b \rangle &= \delta_{ab}(\theta - \theta') G(z,z') \; ; \\
\left\langle \bar C_a \bar D'_b \right\rangle &= \delta_{ab}(\bar\theta - \bar\theta') G(z,z') \; .
\end{aligned}
\tag{4.5}
$$

The brackets on the left hand side of (4.6) indicate that the two-point function has to be calculated with the formula

$$
\langle \ldots \rangle = \int [\mathrm{d}\tilde\mu] \; \ldots \; \exp\left\{ -\frac{1}{e^2} \mathcal{S}_{\mathrm{lin}} \right\} \; . \tag{4.6}
$$

The linearized action $\mathcal{S}_{\mathrm{lin}}$ leads to the equations (4.2), albeit with the source superfields set equal to zero.

4.2 Short Distance Behaviour

GF have singularities in the scalar variable σ, i.e. one fourth of the geodetic interval squared between the points $(z^m, z^{\bar{m}})$ and $(z^{m'}, z^{\bar{m}'})$. The variable satisfies

$$\sigma = g^{\bar{n}m}\sigma_m\sigma_{\bar{n}} \tag{4.7}$$

where

$$\sigma_m = \nabla_m\sigma \; ; \qquad\qquad \sigma_{\bar{m}} = \nabla_{\bar{m}}\sigma \; . \tag{4.8}$$

The residue of the singularity involves the so called parallel displacement matrix $g_{m\bar{n}'}(z,z')$ [38] defined by

$$(\sigma^p\nabla_p + \sigma^{\bar{p}}\nabla_p)\, g_{m\bar{n}'}(z,z') = 0 \; ; \qquad [g_{m\bar{n}'}] = g_{m\bar{n}} \; . \tag{4.9}$$

The square bracket used in the boundary condition means coinciding arguments, i.e. $z^{m'} = z^m$ and $z^{\bar{m}'} = z^{\bar{m}}$.

The scalar Green function has the following short distance behaviour [38]:

$$G(z,z') = -\frac{\Gamma(z,z')}{64\pi^2}\left\{\frac{1}{\sigma} + [v_0]\ln\sigma + \mathcal{O}(\sigma_m\ln\sigma)\right\} \tag{4.10}$$

valid up to terms of at least first order in $\sigma_m\ln\sigma$ or $\sigma_{\bar{m}}\ln\sigma$. The functions $\Gamma(z,z')$ and $v_0(z,z')$ can be determined from differential equations with boundary conditions that are similar to (4.9). The only relation we shall need in the following is:

$$\left(\nabla_m + g_{m\bar{n}'}\nabla^{\bar{n}'}\right)\ln\Gamma(z,z') = \mathcal{O}_2(\sigma_m,\sigma_{\bar{m}}) \; ; \quad [\Gamma] = 1 \tag{4.11}$$

where the symbol $\mathcal{O}_2$ on the right hand side of the first equation (4.11) means terms at least bilinear in σ_m and $\sigma_{\bar{m}}$.

Instead of the vector GF it proves convenient to introduce [39] two bilocal unprimed tensors

$$\begin{aligned} \overline{G}_{m\bar{n}}(z,z') &= g^{\bar{p}'}_{\bar{n}}(z,z')G_{m\bar{p}'}(z,z') \; ; \\ \widetilde{G}_{m\bar{n}}(z,z') &= g^{p'}_{m}(z,z')G_{p'\bar{n}}(z,z') \; . \end{aligned} \tag{4.12}$$

The short distance expansion of the vector GF reads now

$$\overline{G}_{m\bar{n}}(z,z') = g_{m\bar{n}}G(z,z') - \frac{\Gamma(z,z')}{128\pi^2}R_{m\bar{n}}\ln\sigma + \mathcal{O}(\sigma_m\ln\sigma) \tag{4.13}$$

where $R_{m\bar{n}}$ is the Ricci tensor of the Kähler manifold.

4.3 Regularization of Superfield Currents

We assume that the fermionic symmetries are preserved in perturbation theory. The corresponding currents $s^{\pm}_{m}$, $s^{\pm}_{\bar{m}}$ receive radiative corrections, can be, however, redefined as to remain conserved. As a consequence one can use the superspace approach for quantum computations. However, the conservation law (3.17) of the superfield currents $B^{\pm}_{m}$, $B^{\pm}_{\bar{m}}$ is violated in perturbation theory. The breakdown of current conservation gives rise to $\mathbb{R}_{\pm}$-anomalies.

For evaluating the gravitational contribution it is sufficient to compute the vacuum expectation value of the left hand side of the conservation law (3.17). Here we consider explicitly only the $\mathbb{R}_{+}$-anomaly. The appropriate superfield currents were given in (3.27). In the one-loop approximation one uses the linearized expressions

$$\begin{aligned} \widetilde{B}^{+}_{m} &= 4\mathrm{iTr}\Big\{\tfrac{1}{2}g^{\bar{p}n}X_{mn}\partial_{\bar{\theta}}\left(\nabla_{\bar{p}}V+\phi_{\bar{n}}\right)+\partial_{\theta}\partial_{\bar{\theta}}V\nabla_{m}\Lambda \\ &\quad +\mathrm{i}\Big(\nabla_{m}D^{\dagger}\partial_{\bar{\theta}}C^{\dagger}-\nabla_{m}B^{\dagger}\partial_{\bar{\theta}}V\Big)\Big\}\ ; \end{aligned} \tag{4.14}$$

$$\begin{aligned} \widetilde{B}^{+}_{\bar{m}} &= 4\mathrm{iTr}\Big\{\partial_{\bar{\theta}}\left(\nabla_{\bar{m}}V+\phi_{\bar{m}}\right)\partial_{\theta}\Lambda-\Lambda\partial_{\theta}\partial_{\bar{\theta}}\nabla_{\bar{m}}V \\ &\quad -\mathrm{i}\Big(D^{\dagger}\partial_{\bar{\theta}}\nabla_{\bar{m}}C^{\dagger}+B^{\dagger}\partial_{\bar{\theta}}\phi_{\bar{m}}\Big)\Big\}\ . \end{aligned} \tag{4.15}$$

The linearized currents are regularized by symmetric point-splitting being summarized in the following rules:

1. For each term of (4.14) and (4.15) one takes into account both factor orderings with equal weight.

2. Each primed index, i.e. corresponding to the coordinates $(z^{m'}, z^{\bar{m}'})$ is accompanied by a parallel displacement matrix element $g_{m\bar{n}'}(z,z')$ or by its inverse $g^{\bar{n}'m}(z',z)$.

As an example let us write down the regularized version of (4.14)

$$\begin{aligned} \widetilde{B}^{+}_{m}(z,z') &= 2\mathrm{iTr}\Big\{\tfrac{1}{2}\left(g^{\bar{p}'n}X_{mn}\partial_{\bar{\theta}'}\phi_{\bar{p}'}-g^{n'}_{m}g^{\bar{p}r'}\partial_{\bar{\theta}}\phi_{\bar{p}}X_{n'r'}\right) \\ &\quad +\nabla_{m}\Lambda\partial_{\theta'}\partial_{\bar{\theta}'}V'+g^{n'}_{m}\partial_{\theta}\partial_{\bar{\theta}}V\nabla_{n'}\Lambda' \\ &\quad +\mathrm{i}\left(\nabla_{m}D^{\dagger}\partial_{\bar{\theta}'}C'^{\dagger}+g^{n'}_{m}\partial_{\bar{\theta}}C^{\dagger}\nabla_{n'}B'^{\dagger}\right) \\ &\quad -\mathrm{i}\Big(\nabla_{m}B^{\dagger}\partial_{\bar{\theta}'}V'-g^{n'}_{m}\partial_{\bar{\theta}}V\nabla_{n'}B'^{\dagger}\Big)\Big\}\ . \end{aligned} \tag{4.16}$$

One finds a similar expression for $\widetilde{B}^+_{\bar{m}}(z,z')$. The vacuum expectation values of the regularized currents are

$$\begin{aligned}\left\langle \widetilde{B}^+_m(z,z')\right\rangle &= 2n\Big\{g^{\bar{p}n'}\nabla_{[m}G_{n]\bar{p}'} - \nabla_m G \\ &\quad + g^{n'}_m\left(g^{\bar{p}r'}\nabla_{[n'}G_{r']\bar{p}} - \nabla_{n'}G\right)\Big\}\ ; \\ \left\langle \widetilde{B}^+_{\bar{m}}(z,z')\right\rangle &= 0\ .\end{aligned} \tag{4.17}$$

Recalling the meaning of the square bracket in eqs. (4.9) and (4.10) the gravitational contribution to the $\mathbb{R}_+$-anomaly becomes

$$B^{(+)}(z) = \frac{1}{2}\nabla^m\left[\left\langle \widetilde{B}^+_m(z,z')\right\rangle\right] \tag{4.18}$$

can be evaluated with the help of Synge's theorem [39]

$$B^{(+)}(z) = \frac{1}{2}\left[\left\langle\left(\nabla^m + g^{\bar{n}'m}\nabla_{\bar{n}'}\right)\widetilde{B}^+_m(z,z')\right\rangle\right]\ . \tag{4.19}$$

If one inserts (4.17) into (4.19) and if one tries to exhibit the combinations (4.12) in place of the vector GF, one gets

$$\begin{aligned}B^{(+)}(z) &= n\Big[\Big\{\left(g^{\bar{n}p'}\nabla_{[\bar{s}}g_{p'\bar{u}]} - g_{p'[\bar{s}}g_{r'\bar{u}]}\nabla^{p'}g^{\bar{n}r'}\right)\left(\nabla^{\bar{s}}\,\overline{G}^{\bar{u}}_{\bar{n}} + \nabla^{\bar{s}}\widetilde{G}^{\bar{u}}_{\bar{n}}\right. \\ &\quad \left. + g^{\bar{t}'}_{\bar{n}}\nabla^{\bar{s}}g^{\bar{v}}_{\bar{t}'}\overline{G}^{\bar{u}}_{\bar{v}} - g^{\bar{s}t'}g^{\bar{u}v'}\nabla_{t'}g^{w}_{v'}\widetilde{G}_{w\bar{n}}\right) \\ &\quad + \left(\nabla_{\bar{m}'}g^{\bar{m}'}_{\bar{s}} - g_{n'\bar{s}}\nabla_{\bar{m}}g^{\bar{m}n'}\right)\left(2g^{\bar{s}u'}\nabla_{u'}G + \nabla^{[\bar{s}}\widetilde{G}^{\bar{u}]}_{\bar{u}}\right. \\ &\quad \left. -\ g^{[\bar{s}t'}g^{\bar{u}]v'}\nabla_{t'}g^{p}_{v'}\widetilde{G}_{p\bar{u}}\right)\Big\}\Big]\ .\end{aligned} \tag{4.20}$$

Now one uses the short distance expansions (4.10) and (4.13) in (4.20). One realizes immediately that only singularities of the form σ^{-2} contribute to (4.20). The residues can be evaluated by means of the formulae:

$$\begin{aligned}&g^{p'}_n\nabla_{[\bar{s}}g_{p'\bar{u}]} - g_{p'[\bar{s}}g_{r'\bar{u}]}\nabla^{p'}g^{r'}_n \\ &\qquad = \frac{1}{12}\sigma^a\sigma^b\sigma^{\bar{c}}\left(R^r{}_{ab[\bar{s}}R_{\bar{u}]rn\bar{c}} - R^r{}_{an[\bar{s}}R_{\bar{u}]rb\bar{c}}\right) + \ldots \\ &\nabla_{\bar{m}'}g^{\bar{m}'}_{\bar{s}} - g_{n'\bar{s}}\nabla_{\bar{m}}g^{\bar{m}n'} \\ &\qquad = -\frac{1}{6}\sigma^a\sigma^b\sigma^{\bar{c}}\left(\frac{1}{2}R^p{}_{ra\bar{s}}R^r{}_{pb\bar{c}} + \frac{1}{4}R^p{}_{ab\bar{c}}R_{p\bar{s}} + R^p{}_aR_{\bar{c}pb\bar{s}}\right) + \ldots\end{aligned} \tag{4.21}$$

where the dots collect all the terms which are unimportant for the present calculation. The result reads

$$
\begin{aligned}
B^{(+)}(z) \;=\; & \frac{n}{128\pi^2}\left[\frac{\sigma^a\sigma^{\bar b}}{\sigma}\left\{\frac{\sigma^c\sigma^{\bar d}}{3\sigma}\left(R^p{}_{ac\bar b}R_{p\bar d} + R^p{}_a R_{\bar b pc\bar d}\right.\right.\right. \\
& \left.\left.\left. - R^p{}_{ra\bar b}R^r{}_{pc\bar d} - R^p{}_{ac}{}^r R_{\bar b pr\bar d}\right) + R^p{}_a R_{p\bar b} - RR_{a\bar b}\right\}\right] .
\end{aligned} \tag{4.22}
$$

The average over the directions of σ_m, $\sigma_{\bar m}$ amounts to the following replacements:

$$
\sigma^a\sigma^{\bar b}\rightarrow g^{\bar b a}\frac{\sigma}{2} \; ; \qquad \sigma^a\sigma^b\sigma^{\bar c}\sigma^{\bar d}\rightarrow \left(g^{\bar c a}g^{\bar d b} + g^{\bar d a}g^{\bar c b}\right)\frac{\sigma^2}{6} \; . \tag{4.23}
$$

One can avoid the average if one performs the subtraction of certain direction dependent terms [40] in the regularized current.

As a result the gravitational contribution to the $\mathbb{R}_+$-anomaly assumes the manifestly local form

$$
B^{(+)} = \frac{n}{64\pi^2}\left(\frac{1}{3}R^a{}_b R^b{}_a - \frac{1}{4}R^2 - \frac{1}{12}R^a{}_{bc}{}^d R^b{}_{ad}{}^c\right) \; . \tag{4.24}
$$

It turns out that the contribution $B^{(-)}$ of the external gravity to the $\mathbb{R}_-$-anomaly has the same expression.

Before discussing the result (4.24) let us comment on its derivation. The contribution of the Faddeev–Popov ghosts cancels in (4.24), hence the same result is obtained if one keeps only the first three terms in (4.14) and (4.15). The cancellation is due to the special interplay between BRS and fermionic symmetries. suggests that the calculation could be performed without Faddeev–Popov ghosts, but with the ghosts introduced in section 5.1.

Eq. (4.24) can be written in the more familiar form

$$
B^{(\pm)} = n(H - E) \; . \tag{4.25}
$$

Here E and H are the Kähler analogs of the Euler and Hirzebruch (signature) densities:

$$
\begin{aligned}
E &= \frac{1}{128\pi^2}\left(R^m{}_{pr}{}^n R^p{}_{mn}{}^r - 2R^m{}_n R^n{}_m + R^2\right) \; ; \\
H &= \frac{1}{192\pi^2}\left(R^m{}_{pr}{}^n R^p{}_{mn}{}^r - R^m{}_n R^n{}_m\right) \; .
\end{aligned} \tag{4.26}
$$

Recall that on a Riemannian manifold without boundary, the densities (4.26) have the expressions [41]

$$E = \frac{1}{32\pi^2}\widetilde{R}_{\lambda\mu\nu\rho}\widetilde{R}^{\nu\rho\lambda\mu}\ ; \qquad H = \frac{1}{48\pi^2}\widetilde{R}_{\lambda\mu\nu\rho}R^{\nu\rho\lambda\mu} \tag{4.27}$$

where $R_{\lambda\mu\nu\rho}$ is the curvature tensor and $\widetilde{R}_{\lambda\mu\nu\rho}$ its dual

$$\widetilde{R}_{\lambda\mu\nu\rho} = \frac{1}{2}\sqrt{g}\,\epsilon_{\lambda\mu\alpha\beta}R_{\nu\rho}{}^{\alpha\beta}\ . \tag{4.28}$$

One can write the curvature tensor $R_{\lambda\mu\nu\rho}$ in spinorial form and decompose it into irreducible spinors [31], [42]. In passing to Kähler four-manifolds one realizes that the curvature tensor has the structure $R_{\bar{m}np\bar{r}}$, symmetric in $\bar{m}$, $\bar{r}$ and n, p respectively. In terms of irreducible spinors it has the form

$$\begin{aligned} R_{\bar{m}pr\bar{n}} \;=\; & 4\Big\{2\left(g_{p\bar{m}}g_{r\bar{n}} + g_{r\bar{m}}g_{p\bar{n}}\right)U - e^{\alpha}_{\bar{m}}e^{\beta}_{p}e^{\gamma}_{r}e^{\delta}_{\bar{n}}U_{\alpha\beta\gamma\delta} \\ & + \left(e^{\alpha}_{\bar{m}}e^{\beta}_{p}g_{r\bar{n}} + e^{\alpha}_{\bar{n}}e^{\beta}_{r}g_{p\bar{m}}\right)U_{\alpha\beta}\Big\}\ . \end{aligned} \tag{4.29}$$

Here U, $U_{\alpha\beta}$ and $U_{\alpha\beta\gamma\delta}$ are irreducible spinors, completely symmetric in their indices. The zweibeins e^{α}_{m}, $e^{\beta}_{\bar{n}}$ convert spinor indices into holomorphic and anti-holomorphic ones.

The densities (4.26) can also be written in terms of irreducible spinors. For Kähler manifolds one finds:

$$\begin{aligned} E \;&=\; \frac{1}{8\pi^2}\left(U_{\alpha\beta\gamma\delta}U^{\alpha\beta\gamma\delta} + 4U_{\alpha\beta}U^{\alpha\beta} + 48U^2\right)\ ; \\ H \;&=\; \frac{1}{12\pi^2}\left(U_{\alpha\beta\gamma\delta}U^{\alpha\beta\gamma\delta} - 24U^2\right)\ . \end{aligned} \tag{4.30}$$

By using (4.29) it is possible to express (4.30) through the curvature tensor over the Kähler manifold. The result is (4.26).

5 Donaldson Cohomology

Let $(\mathcal{K}, \gamma)$ be a compact Kähler four-manifold with Kähler form γ given by (2.1). Let E be a complex vector bundle with structure group G assumed Lie, compact and semi-simple. The connection on E splits into a $(1,0)$ part $a = a_m \mathrm{d}z^m$ and

a $(0,1)$ part $\bar{a} = a_{\bar{m}} \mathrm{d}z^{\bar{m}}$. The curvature two-form can be decomposed into its $(2,0)$, $(0,2)$ and $(1,1)$ parts as follows:

$$
\begin{gathered}
f^{(2,0)} = \partial a + a^2 \; ; \qquad f^{(0,2)} = \bar{\partial}\bar{a} + \bar{a}^2 \; ; \\
f^{(1,1)} = \partial \bar{a} + \bar{\partial} a + \{a, \bar{a}\} \; .
\end{gathered}
\tag{5.1}
$$

They obey the Bianchi identities

$$
\begin{gathered}
\mathrm{D} f^{(1,1)} + \bar{\mathrm{D}} f^{(2,0)} = 0 \; ; \qquad \bar{\mathrm{D}} f^{(1,1)} + \mathrm{D} f^{(0,2)} = 0 \; ; \\
\mathrm{D} f^{(2,0)} = \bar{\mathrm{D}} f^{(0,2)} = 0 \; .
\end{gathered}
\tag{5.2}
$$

By using (5.2) one derives the basic identities (For simplicity we limit ourselves to invariant polynomials quadratic in the curvature)

$$
\begin{aligned}
\partial \; \mathrm{Tr} \left(\tfrac{1}{2} {f^{(1,1)}}^2 + f^{(2,0)} f^{(0,2)} \right) + \bar{\partial} \; \mathrm{Tr} f^{(1,1)} f^{(2,0)} &= 0 \; ; \\
\bar{\partial} \; \mathrm{Tr} \left(\tfrac{1}{2} {f^{(1,1)}}^2 + f^{(2,0)} f^{(0,2)} \right) + \partial \; \mathrm{Tr} f^{(1,1)} f^{(0,2)} &= 0 \; .
\end{aligned}
\tag{5.3}
$$

Obviously, the last terms in both eqs. (5.3) vanish, rendering the invariant $(2,2)$-form $\mathrm{Tr} \left(\frac{1}{2} f^{(1,1)\,2} + f^{(2,0)} f^{(0,2)} \right)$ closed with respect to both ∂ and $\bar{\partial}$.

Locally, the closed form can be represented as

$$
\mathrm{Tr} \left(\tfrac{1}{2} {f^{(1,1)}}^2 + f^{(2,0)} f^{(0,2)} \right) = \bar{\partial} K + \partial \bar{K}
\tag{5.4}
$$

where

$$
\begin{aligned}
K &= \tfrac{1}{2} \mathrm{Tr} \left(\bar{a} \partial a + a f^{(1,1)} \right) \; ; \\
\bar{K} &= \tfrac{1}{2} \mathrm{Tr} \left(a \bar{\partial} \bar{a} + \bar{a} f^{(1,1)} \right) \; .
\end{aligned}
\tag{5.5}
$$

One can easily check that $\partial\bar{\partial} K = \partial\bar{\partial}\bar{K} = 0$. While eqs. (5.5) render the cohomology of ∂, $\bar{\partial}$ trivial, they are not gauge invariant.

5.1 Descent Equations and Their Solution

The fermionic symmetries q, $\bar{\mathrm{q}}$ act as follows

$$
\begin{array}{rclcrcl}
\mathrm{q}a &=& \psi - \mathrm{D}\omega\ ; & & \bar{\mathrm{q}}a &=& -\mathrm{D}\bar{\omega}\ ; \\
\mathrm{q}\bar{a} &=& -\bar{\mathrm{D}}\omega\ ; & & \bar{\mathrm{q}}\bar{a} &=& \bar{\psi} - \bar{\mathrm{D}}\bar{\omega}\ ; \\
\mathrm{q}\psi &=& [\psi,\omega]\ ; & & \bar{\mathrm{q}}\psi &=& -\mathrm{i}\mathrm{D}\varphi + [\psi,\bar{\omega}]\ ; \\
\mathrm{q}\bar{\psi} &=& -\mathrm{i}\bar{\mathrm{D}}\varphi + [\bar{\psi},\omega]\ ; & & \bar{\mathrm{q}}\bar{\psi} &=& [\bar{\psi},\bar{\omega}]\ ; \\
\mathrm{q}\omega &=& -\omega^2\ ; & & \bar{\mathrm{q}}\bar{\omega} &=& -\bar{\omega}^2\ ; \\
\mathrm{q}\varphi &=& [\varphi,\omega]\ ; & & \bar{\mathrm{q}}\varphi &=& [\varphi,\bar{\omega}]\ ; \\
\mathrm{i}\varphi &=& \mathrm{q}\bar{\omega} + \bar{\mathrm{q}}\omega + \{\omega,\bar{\omega}\}\ . & & & &
\end{array}
\tag{5.6}
$$

The ghosts ω, $\bar{\omega}$ are the first components of the Grassmann superconnections A_θ and $A_{\bar{\theta}}$ respectively. They occur as supergauge transformations in superspace (see section 2.2) and ensure the nilpotency and anticommutativity of the fermionic symmetries

$$
\mathrm{q}^2 = \bar{\mathrm{q}}^2 = \mathrm{q}\bar{\mathrm{q}} + \bar{\mathrm{q}}\mathrm{q} = 0\ . \tag{5.7}
$$

Here a comment is in order, since apparently one cannot separate the action of q on $\bar{\omega}$ from that of $\bar{\mathrm{q}}$ on ω. In fact, the ghosts ω, $\bar{\omega}$ can be expressed through the same prepotential V, as discussed in section 2.4. Of course, V is determined up to chiral gauge transformations, i.e. up to local parameters which are annihilated either by q or by $\bar{\mathrm{q}}$.

The procedure we shall now describe is an extension of the construction [43], [44], [45] to complex manifolds. Let $\mathcal{A}$ be the space of all connections on the complex vector bundle E and $\mathcal{G}$ be the group of gauge transformations. The quotient $\mathcal{B} = \mathcal{A}\backslash\mathcal{G}$ is the set of all gauge equivalent connections. Replace ∂ by $\Delta = \partial + \bar{\mathrm{q}}$ and $\bar{\partial}$ by $\bar{\Delta} = \bar{\partial} + \mathrm{q}$. The derivations Δ, $\bar{\Delta}$ act over the product space $\mathcal{K}\times\mathcal{B}$ and satisfy

$$
\Delta^2 = \bar{\Delta}^2 = \Delta\bar{\Delta} + \bar{\Delta}\Delta = 0\ . \tag{5.8}
$$

Furthermore, one defines an adjoint bundle $\mathcal{E}$ over $\mathcal{K}\times\mathcal{B}$. Let $\mathcal{A} = a + \bar{\omega}$ and $\bar{\mathcal{A}} = \bar{a} + \omega$ be the connections on $\mathcal{E}$. One can construct generalized forms for the curvature

$$
\begin{gathered}
\mathcal{F}^{(2,0)} = \Delta\mathcal{A} + \mathcal{A}^2\ ; \qquad \mathcal{F}^{(0,2)} = \bar{\Delta}\bar{\mathcal{A}} + \bar{\mathcal{A}}^2\ ; \\
\mathcal{F}^{(1,1)} = \Delta\bar{\mathcal{A}} + \bar{\Delta}\mathcal{A} + \{\mathcal{A},\bar{\mathcal{A}}\}\ .
\end{gathered}
\tag{5.9}
$$

The quantities (5.9) satisfy Bianchi identities similar to (5.2)

$$\begin{gathered}\Delta\mathcal{F}^{(2,0)}+[\mathcal{A},\mathcal{F}^{(2,0)}]=0\ ;\qquad \bar\Delta\mathcal{F}^{(0,2)}+[\bar{\mathcal{A}},\mathcal{F}^{(0,2)}]=0\ ;\\ \Delta\mathcal{F}^{(1,1)}+\bar\Delta\mathcal{F}^{(2,0)}+[\mathcal{A},\mathcal{F}^{(1,1)}]+[\bar{\mathcal{A}},\mathcal{F}^{(2,0)}]=0\ ;\\ \Delta\mathcal{F}^{(0,2)}+\bar\Delta\mathcal{F}^{(1,1)}+[\mathcal{A},\mathcal{F}^{(0,2)}]+[\bar{\mathcal{A}},\mathcal{F}^{(1,1)}]=0\ .\end{gathered}\tag{5.10}$$

Also basic identities look similar to (5.3):

$$\begin{aligned}\Delta\ \mathrm{Tr}\left(\tfrac{1}{2}\mathcal{F}^{(1,1)\ 2}+\mathcal{F}^{(2,0)}\mathcal{F}^{(0,2)}\right)+\bar\Delta\ \mathrm{Tr}\ \mathcal{F}^{(1,1)}\mathcal{F}^{(2,0)}&=0\ ;\\ \bar\Delta\ \mathrm{Tr}\left(\tfrac{1}{2}\mathcal{F}^{(1,1)\ 2}+\mathcal{F}^{(2,0)}\mathcal{F}^{(0,2)}\right)+\Delta\ \mathrm{Tr}\ \mathcal{F}^{(1,1)}\mathcal{F}^{(0,2)}&=0\ .\end{aligned}\tag{5.11}$$

However, the last terms of (5.10) do not vanish, since

$$\mathrm{Tr}\ \mathcal{F}^{(1,1)}\mathcal{F}^{(2,0)}=\mathrm{Tr}f^{(2,0)}\left(\bar\psi+\mathrm{i}\varphi\right)\ ;\ \ \mathrm{Tr}\ \mathcal{F}^{(1,1)}\mathcal{F}^{(0,2)}=\mathrm{Tr}f^{(0,2)}\left(\psi+\mathrm{i}\varphi\right)\ .\tag{5.12}$$

Due to Bianchi identities (5.10) the expressions (5.11) are Δ-close and $\bar\Delta$-close respectively

$$\Delta\mathrm{Tr}\mathcal{F}^{(1,1)}\mathcal{F}^{(2,0)}=0\ ;\qquad \bar\Delta\mathrm{Tr}\mathcal{F}^{(1,1)}\mathcal{F}^{(0,2)}=0\ .\tag{5.13}$$

By enlarging the field manifold one can make them Δ and $\bar\Delta$ exact. This feature makes the theory of Donaldson polynomials somewhat different from that of TYM with a single fermionic symmetry.

Let us introduce the forms χ, b and $\bar\chi$, $\bar b$ of $(2,0)$ and $(0,2)$ type, respectively. They obey

$$\begin{aligned}\mathrm{q}\chi&=-\mathrm{i}b-\{\omega,\chi\}\ ; & \bar{\mathrm{q}}\chi&=-\mathrm{i}f^{(2,0)}-\{\bar\omega,\chi\}\ ;\\ \mathrm{q}\bar\chi&=-\mathrm{i}f^{(0,2)}-\{\omega,\bar\chi\}\ ; & \bar{\mathrm{q}}\bar\chi&=-\mathrm{i}\bar b-\{\bar\omega,\bar\chi\}\ ;\\ \mathrm{q}b&=[b,\omega]\ ; & \bar{\mathrm{q}}b&=\mathrm{D}\psi-[\varphi,\chi]+[b,\bar\omega]\ ;\\ \mathrm{q}\bar b&=\bar{\mathrm{D}}\bar\psi-[\varphi,\bar\chi]+[\bar b,\omega]\ ; & \bar{\mathrm{q}}\bar b&=[\bar b,\bar\omega]\ .\end{aligned}\tag{5.14}$$

One can check that

$$\begin{aligned}\mathrm{Tr}\ \mathcal{F}^{(1,1)}\mathcal{F}^{(2,0)}&=\mathrm{i}\bar{\mathrm{q}}\ \mathrm{Tr}\chi\left(\bar\psi+\mathrm{i}\varphi\right)=\mathrm{i}\Delta\ \mathrm{Tr}\chi\left(\bar\psi+\mathrm{i}\varphi\right)\ ;\\ \mathrm{Tr}\ \mathcal{F}^{(1,1)}\mathcal{F}^{(0,2)}&=\mathrm{i}\mathrm{q}\ \mathrm{Tr}\bar\chi\left(\psi+\mathrm{i}\varphi\right)=\mathrm{i}\bar\Delta\ \mathrm{Tr}\bar\chi\left(\psi+\mathrm{i}\varphi\right)\ .\end{aligned}\tag{5.15}$$

By inserting (5.14) into (5.10) one gets

$$
\begin{aligned}
\Delta \mathrm{Tr}\left(\tfrac{1}{2}\mathcal{F}^{(1,1)^2}+\mathcal{F}^{(2,0)}\mathcal{F}^{(0,2)}\right)-\mathrm{i}\bar{\Delta}\mathrm{Tr}\chi\left(\bar{\psi}+\mathrm{i}\varphi\right) &= 0\ ; \\
\bar{\Delta} \mathrm{Tr}\left(\tfrac{1}{2}\mathcal{F}^{(1,1)^2}+\mathcal{F}^{(2,0)}\mathcal{F}^{(0,2)}\right)-\mathrm{i}\Delta\mathrm{Tr}\bar{\chi}\left(\psi+\mathrm{i}\varphi\right) &= 0\ .
\end{aligned}
\tag{5.16}
$$

Let us now define

$$
\begin{aligned}
\mathcal{W} = & -\frac{c}{2\pi^2}\Big\{\mathrm{Tr}\left(\tfrac{1}{2}\mathcal{F}^{(1,1)\ 2}+\mathcal{F}^{(2,0)}\mathcal{F}^{(0,2)}\right) \\
& -\mathrm{i}\Delta\ \ \mathrm{Tr}\bar{\chi}\left(\psi+\mathrm{i}\varphi\right)-\mathrm{i}\bar{\Delta}\ \mathrm{Tr}\chi\left(\bar{\psi}+\mathrm{i}\varphi\right)\Big\}\ .
\end{aligned}
\tag{5.17}
$$

Here the factor in front is conventional, and c is the second Casimir invariant of the adjoint representation of the $\mathcal{G}$, i.e. $c_{acd}c_{bcd}=c\delta_{ab}$. Eqs. (5.15) take the form

$$
\Delta\mathcal{W}=0\ ;\qquad \bar{\Delta}\mathcal{W}=0\ . \tag{5.18}
$$

By expanding $\mathcal{W}$ in the ghost numbers associated to $\mathbb{R}_\pm$ (lower bracket) one gets the descent equations:

$$
\begin{aligned}
\mathrm{q}W^{(2,2)}_{(0,0)} &= -\bar{\partial}W^{(2,1)}_{(1,0)}\ ; & \bar{\mathrm{q}}W^{(2,2)}_{(0,0)} &= -\partial W^{(1,2)}_{(0,1)}\ ; \\
\mathrm{q}W^{(2,1)}_{(1,0)} &= -\bar{\partial}W^{(2,0)}_{(2,0)}\ ; & \bar{\mathrm{q}}W^{(2,1)}_{(1,0)} &= -\partial W^{(1,1)}_{(1,1)}\ ; \\
\mathrm{q}W^{(1,2)}_{(0,1)} &= -\bar{\partial}W^{(1,1)}_{(1,1)}\ ; & \bar{\mathrm{q}}W^{(1,2)}_{(0,1)} &= -\partial W^{(0,2)}_{(0,2)}\ ; \\
\mathrm{q}W^{(1,1)}_{(1,1)} &= -\bar{\partial}W^{(1,0)}_{(2,1)}\ ; & \bar{\mathrm{q}}W^{(1,1)}_{(1,1)} &= -\partial W^{(0,1)}_{(2,1)}\ ; \\
\mathrm{q}W^{(2,0)}_{(2,0)} &= 0\ ; & \bar{\mathrm{q}}W^{(2,0)}_{(2,0)} &= -\partial W^{(1,0)}_{(2,1)}\ ; \\
\mathrm{q}W^{(0,2)}_{(0,2)} &= -\bar{\partial}W^{(0,1)}_{(1,2)}\ ; & \bar{\mathrm{q}}W^{(0,2)}_{(0,2)} &= 0\ ; \\
\mathrm{q}W^{(1,0)}_{(1,2)} &= 0\ ; & \bar{\mathrm{q}}W^{(1,0)}_{(2,1)} &= -\partial W^{(0,0)}_{(2,2)}\ ; \\
\mathrm{q}W^{(0,1)}_{(1,2)} &= -\bar{\partial}W^{(0,0)}_{(2,2)}\ ; & \bar{\mathrm{q}}W^{(0,1)}_{(1,2)} &= 0\ ; \\
\mathrm{q}W^{(0,0)}_{(0,0)} &= 0\ ; & \bar{\mathrm{q}}W^{(0,0)}_{(2,2)} &= 0\ .
\end{aligned}
\tag{5.19}
$$

The upper bracket indicates the type of form on Kähler manifold.

In order to obtain the solution of (5.19) we write (5.17) in the form

$$
\begin{aligned}
\mathcal{W} &= -\frac{c}{4\pi^2}\,\mathrm{Tr}\left(f^{(1,1)}+f^{(2,0)}+f^{(0,2)}+\psi+\bar{\psi}+\mathrm{i}\varphi\right)^2 \\
&\quad+\frac{\mathrm{i}c}{2\pi^2}\left\{\Delta\ \mathrm{Tr}\bar{\chi}\left(\psi+\mathrm{i}\varphi\right)+\bar{\Delta}\ \mathrm{Tr}\chi\left(\bar{\psi}+\mathrm{i}\varphi\right)\right\}\ .
\end{aligned}
\tag{5.20}
$$

By expanding $\mathcal{W}$ according to $r_\pm$ numbers one gets

$$
\begin{aligned}
W^{(2,2)}_{(0,0)} &= \frac{c}{4\pi^2}\mathrm{Tr}\left\{\tfrac{1}{2}f^{(1,1)^2}+f^{(2,0)}f^{(0,2)}-\mathrm{i}\partial\left(\bar{\chi}\psi\right)-\mathrm{i}\bar{\partial}\left(\chi\bar{\psi}\right)\right\}\ ; \\
W^{(2,1)}_{(1,0)} &= \frac{c}{4\pi^2}\mathrm{Tr}\left(\varphi\bar{\mathrm{D}}\chi-f^{(1,1)}\psi-b\bar{\psi}\right)\ ; \\
W^{(1,2)}_{(0,1)} &= \frac{c}{4\pi^2}\mathrm{Tr}\left(\varphi\mathrm{D}\bar{\chi}-f^{(1,1)}\bar{\psi}-\bar{b}\psi\right)\ ; \\
W^{(2,0)}_{(2,0)} &= \frac{c}{4\pi^2}\mathrm{Tr}\left(\tfrac{1}{2}\psi^2-\mathrm{i}\varphi b\right)\ ; \qquad W^{(0,2)}_{(0,2)} = \frac{c}{4\pi^2}\mathrm{Tr}\left(\tfrac{1}{2}\bar{\psi}^2-\mathrm{i}\varphi\bar{b}\right)\ ; \\
W^{(1,1)}_{(1,1)} &= \frac{c}{4\pi^2}\mathrm{Tr}\left(\mathrm{i}\varphi f^{(1,1)}+\psi\bar{\psi}\right)\ ; \quad W^{(1,0)}_{(2,1)} = \frac{\mathrm{i}c}{4\pi^2}\mathrm{Tr}\varphi\psi\ ; \\
W^{(0,1)}_{(1,2)} &= \frac{\mathrm{i}c}{4\pi^2}\mathrm{Tr}\varphi\bar{\psi}\ ; \qquad\qquad W^{(0,0)}_{(2,2)} = -\frac{c}{8\pi^2}\mathrm{Tr}\varphi^2\ .
\end{aligned}
\tag{5.21}
$$

The numbers in brackets can be checked by using their additivity as well as table 1. The meaning of the letters in the first column is the following: r_+, r_- are the quantum numbers of the global Abelian symmetries, (p,q) is the complex form degree and d the canonical dimension of the field.

One can show further that

$$
W^{(0,0)}_{(2,2)} = \frac{c}{8\pi^2}\left\{\mathrm{q}\mathrm{Tr}\left(\mathrm{i}\varphi-\omega\bar{\omega}\right)\bar{\omega}+\bar{\mathrm{q}}\mathrm{Tr}\left(\mathrm{i}\varphi+\bar{\omega}\omega\right)\omega\right\}\ . \tag{5.22}
$$

Nevertheless, $W^{(0,0)}_{(2,2)}$ is a nontrivial element of the (equivariant) cohomology of q and $\bar{\mathrm{q}}$, since it does not depend on ω or $\bar{\omega}$. In other words, both $\mathrm{Tr}(\mathrm{i}\varphi-\omega\bar{\omega})\bar{\omega}$ and $\mathrm{Tr}(\mathrm{i}\varphi+\bar{\omega}\omega)\omega$ are not gauge invariant.

Hence, $W^{(p,q)}_{(r_+,r_-)}$ are local observables whose correlation functions might be nonvanishing. Of course, the $W^{(p,q)}_{(r_+,r_-)}$ are gauge invariant, i.e. $\mathrm{s}W^{(p,q)}_{(r_+,r_-)}=0$. Notice that s, q and $\bar{\mathrm{q}}$ anticommute with each other.

	a	$\bar{a}$	ψ	$\bar{\psi}$	χ	$\bar{\chi}$	b	$\bar{b}$	φ	λ	g_+	g_-	k
r_+	0	0	1	0	0	-1	1	-1	1	-1	0	-1	0
r_-	0	0	0	1	-1	0	-1	1	1	-1	-1	0	0
p	1	0	1	0	2	0	2	0	0	0	0	0	0
q	0	1	0	1	0	2	0	2	0	0	0	0	0
d	1	1	1	1	2	2	2	2	0	2	2	2	2

Table 1: Quantum numbers and form degrees of various fields.

5.2 Cohomology Classes

Let us consider the equivalence classes of (p,q)-forms which are both ∂ and $\bar{\partial}$ closed but not exact. A (p,q)-form is exact if either $\omega_{(p,q)} = \partial\bar{\partial}\phi_{(p-1,q-1)}$ or $\omega_{(p,0)} = \partial\phi_{p-1}(z)$ or else $\omega_{(0,q)} = \bar{\partial}\bar{\phi}_{q-1}(\bar{z})$, respectively; here $p,q\geq 1$. The equivalence classes make up a vector space known as the Dolbeault cohomology group $\mathcal{H}^{(p,q)}(\mathcal{K};\partial,\bar{\partial})$. (The composition law is the additive group structure of the vector space.) Let us define

$$\Omega_{(r_+,r_-)} = \int_{\mathcal{K}} W^{(2-p,2-q)}_{(r_+,r_-)}\omega_{(p,q)} \tag{5.23}$$

where $\omega_{(p,q)}$ is a ∂ and $\bar{\partial}$ closed (p,q)-form independent of the fields. One can check that (5.23) is annihilated by both fermionic charges q, $\bar{\mathrm{q}}$. Furthermore, if $\omega_{(p,q)}$ is exact, then $\Omega_{(r_+=q,r_-=p)}$ is highest component, i.e. it can be written in one of the following ways: $\mathrm{q}\bar{\mathrm{q}}\Phi_{(q-1,p-1)}$, $\mathrm{q}\Phi_{(q-1,0)}$ or $\bar{\mathrm{q}}\Phi_{(0,p-1)}$.

We shall see in the next subsection that q, $\bar{\mathrm{q}}$ can be interpreted as complex exterior derivatives on the instanton moduli space $\mathcal{M}$. Since $\Omega_{(q,p)}$ is a (q,p)-form closed with respect to both q and $\bar{\mathrm{q}}$, it belongs to the Dolbeault group $\mathcal{H}^{(q,p)}(\mathcal{M};\mathrm{q},\bar{\mathrm{q}})$. The Donaldson map between the Dolbeault cohomology groups relates $\omega_{(p,q)} \in \mathcal{H}^{(p,q)}(\mathcal{K};\partial,\bar{\partial})$ to $\Omega_{(q,p)} \in \mathcal{H}^{(q,p)}(\mathcal{M};\mathrm{q},\bar{\mathrm{q}})$.

5.3 Donaldson Invariants

For computing invariant correlation functions one needs the integrated observables (5.23). It is convenient to express them in the form:

$$
\begin{aligned}
\Omega_{(0,0)} &= \frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left(\tfrac{1}{2}{f^{(1,1)}}^2 + f^{(2,0)}f^{(0,2)}\right) \ ; \\
\Omega_{(1,0)} &= -\frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left\{f^{(1,1)}\psi + \mathrm{i}\mathrm{q}\left(\chi\bar{\psi}\right)\right\}\omega_{(0,1)} \ ; \\
\Omega_{(0,1)} &= -\frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left\{f^{(1,1)}\bar{\psi} + \mathrm{i}\bar{\mathrm{q}}\left(\bar{\chi}\psi\right)\right\}\omega_{(1,0)} \ ; \\
\Omega_{(2,0)} &= \frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left\{\tfrac{1}{2}\psi^2 + \mathrm{q}\left(\chi\varphi\right)\right\}\omega_{(0,2)} \ ; \\
\Omega_{(0,2)} &= \frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left\{\tfrac{1}{2}\bar{\psi}^2 + \bar{\mathrm{q}}\left(\bar{\chi}\varphi\right)\right\}\omega_{(2,0)} \ ; \\
\Omega_{(1,1)} &= \frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left(\mathrm{i}\varphi f^{(1,1)} + \psi\bar{\psi}\right)\omega_{(1,1)} \ ; \\
\Omega_{(2,1)} &= \frac{\mathrm{i}c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\varphi\psi\omega_{(1,2)} \ ; \qquad \Omega_{(1,2)} = \frac{\mathrm{i}c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\varphi\bar{\psi}\omega_{(2,1)} \ ; \\
\Omega_{(2,2)} &= -\frac{c}{8\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\varphi^2\omega_{(2,2)} \ .
\end{aligned}
\tag{5.24}
$$

The correlation functions of the observables (5.24) are the well-known Donaldson invariants and have the form

$$
\left\langle \prod_i \Omega_{(p_i,q_i)} \right\rangle = \int [\mathrm{d}\mu] \prod_i \Omega_{(p_i,q_i)} \ \exp\left\{-\frac{1}{e^2}\mathcal{S}\right\} \ . \tag{5.25}
$$

where the functional integral is performed upon the fields μ figuring in table 1. Since the observables $\Omega_{(p,0)}$, $\Omega_{(0,q)}$ depend on χ, $\bar{\chi}$, a direct integration of the non-zero modes in the path integral is not possible. Nevertheless, one can show that the correlation functions (5.25) remain unchanged if the equations (5.24) are replaced by another system of observables depending only upon the gauge fields and their various topological ghosts. The new observables – denoted by $\widetilde{\Omega}_{(p,q)}$ – are obtained from $\Omega_{(p,q)}$ by interchanging q and $\bar{\mathrm{q}}$. A simple calculation leads to

$$
\begin{aligned}
\widetilde{\Omega}_{(1,0)} &= -\frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left\{f^{(1,1)}\psi + f^{(2,0)}\bar{\psi}\right\}\omega_{(0,1)} \ ; \\
\widetilde{\Omega}_{(0,1)} &= -\frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left\{f^{(1,1)}\bar{\psi} + f^{(0,2)}\psi\right\}\omega_{(1,0)} \ ;
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
\widetilde{\Omega}_{(2,0)} &= \frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left\{\tfrac{1}{2}\psi^2 - \mathrm{i}f^{(2,0)}\varphi\right\}\omega_{(0,2)}\ ; \\
\widetilde{\Omega}_{(0,2)} &= \frac{c}{4\pi^2}\int_{\mathcal{K}} \mathrm{Tr}\left\{\tfrac{1}{2}\bar{\psi}^2 - \mathrm{i}f^{(0,2)}\varphi\right\}\omega_{(2,0)}\ ,
\end{aligned}
$$

while the other observables obviously are not affected.

For the proof let us write the generating functional of (5.25) in the form (α_q, β_p are arbitrary numbers)

$$\left\langle \exp\left\{\sum_q \alpha_q\left(A_q + \bar{\mathrm{q}}B_q\right) + \sum_p \beta_p\left(C_p + \mathrm{q}D_p\right)\right\}\right\rangle \tag{5.27}$$

where $A_q + \bar{\mathrm{q}}B_q$, $C_p + \mathrm{q}D_p$ are the Donaldson integrated observables (5.24). Being invariant under both q and $\bar{\mathrm{q}}$ they obey

$$\bar{\mathrm{q}}A_q = \mathrm{q}C_p = 0\ ; \quad \mathrm{q}A_q = \bar{\mathrm{q}}\mathrm{q}B_q\ ; \quad \bar{\mathrm{q}}C_p = \mathrm{q}\bar{\mathrm{q}}D_p\ . \tag{5.28}$$

Of course, for some q or p one can have $B_q = 0$ or $D_p = 0$.

Let us deform (5.27) continuosly into the generating functional of the new observables $\widetilde{\Omega}_{(p,q)}$ by means of

$$\left\langle \exp\left\{\sum_q \alpha_q\left[A_q + \left((1-u)\bar{\mathrm{q}} + u\mathrm{q}\right)B_q\right] + \sum_p \beta_p\left[C_p + \left((1-u)\mathrm{q} + u\bar{\mathrm{q}}\right)D_p\right]\right\}\right\rangle \tag{5.29}$$

where u is a real parameter in the interval $[0,1]$. By differentiating with respect to u and by using the conditions (5.28) one can show that (5.29) does not depend on u. The generating functional of the new observables is obtained from (5.29) by taking $u = 1$. Due to the u independence it coincides, however, with (5.27). The integrand of any correlation function has been transformed into an expression depending only on the variables a, $\bar{a}$, their topological ghosts ψ, $\bar{\psi}$ and the ghost for ghosts φ. Hence we proved

$$\left\langle \prod_i \Omega_{(p_i,q_i)}\right\rangle = \left\langle \prod_i \widetilde{\Omega}_{(p_i,q_i)}\right\rangle\ . \tag{5.30}$$

The evaluation of the correlation function proceeds along the same lines as for TYM with a single fermionic symmetry [1], by taking advantage of working with the action

$$\frac{1}{8}\mathrm{q}\bar{\mathrm{q}}\int_{\mathcal{K}} \mathrm{Tr}\left(\chi\bar{\chi} - \mathrm{i}\gamma^2\lambda f\right) \tag{5.31}$$

which differs from (2.10) by a q$\bar{\text{q}}$-term. One can now integrate out all non-zero modes. It is usually assumed that there are no zero modes in the variables χ, $\bar{\chi}$, b and $\bar{b}$.

There is a special prescription for handling the ghost for ghosts: The field φ has to be replaced by the solution $\langle\varphi\rangle$ of the differential equation [29]

$$g^{\bar{n}m}\left(\{\mathrm{D}_m, \mathrm{D}_{\bar{n}}\}\langle\varphi\rangle + 2\mathrm{i}\{\psi_m, \psi_{\bar{n}}\}\right) = 0 \tag{5.32}$$

where ψ_m, $\psi_{\bar{m}}$ are the zero modes of the topological ghosts. In solving eq. (5.32) one can meet zero modes of φ, for which the procedure of ref. [13] should be extended to the Kähler case. For simplicity we shall assume in the following that also such zero modes are absent.

The path integral measure takes its canonical form $[\mathrm{d}a][\mathrm{d}\bar{a}][\mathrm{d}\psi][\mathrm{d}\bar{\psi}]$ where a and $\bar{a}$ are solutions of the self-duality conditions:

$$f_{mn} = f_{\bar{m}\bar{n}} = g^{\bar{n}m} f_{m\bar{n}} = 0\ . \tag{5.33}$$

Since ψ and $\bar{\psi}$ are the zero modes of the topological ghosts, they obey the following equations of motion:

$$\mathrm{D}_{[m}\psi_{n]} = \mathrm{D}_{[\bar{m}}\psi_{\bar{n}]} = \mathrm{D}_m\psi^m = \mathrm{D}^m\psi_m = 0\ . \tag{5.34}$$

One can show [1], [29] that instanton deformations orthogonal to purely gauge transformations obey identical equations. Hence ψ and $\bar{\psi}$ are tangent vectors to the instanton moduli space $\mathcal{M}$. Since q, $\bar{\text{q}}$ relate a, $\bar{a}$ to ψ, $\bar{\psi}$, they play the role of exterior derivatives on $\mathcal{M}$.

The integration of ψ, $\bar{\psi}$ is straightforward and transforms the integrand into a wedge product of (p_i, q_i)-forms over the moduli space

$$\left\langle \prod_i \Omega_{(p_i,q_i)} \right\rangle = \int_{\mathcal{M}} \prod_i \Phi_{(p_i,q_i)}\ . \tag{5.35}$$

In writing down eq. (5.35) we assumed that $\mathcal{M}$ can be considered a finite dimensional Kähler manifold [46].

One can now establish a selection rule for the correlation functions as given by (5.35). The action $\mathcal{S}$ needed for computing the left hand side is invariant under the global Abelian symmetry $\mathbb{R}_+ \otimes \mathbb{R}_-$. In contrast the integration measure transforms under $\mathbb{R}_\pm$ with certain weights that are equal and exactly compensate

the dimension of $\mathcal{M}$. Therefore $\Omega_{(p_i,q_i)}$ should provide the compensating total weights

$$\sum_i p_i = \sum_i q_i = \dim \mathcal{M} \ . \tag{5.36}$$

This means that the integrand of (5.35) is a top-form, i.e. a $(\dim \mathcal{M}, \dim \mathcal{M})$-form over $\mathcal{M}$.

The careful reader may have noticed that the correlation functions were defined by using the action (2.10) in which the BRS gauge fixing has been neglected. The importance of the BRS gauge fixing conditions both for interpreting and computing Donaldson invariants has been emphasized for TYM with a single fermionic charge in [12].

In our case the BRS gauge fixing appears in the total action $\widetilde{\mathcal{S}}+\mathcal{S}'$ suggesting its use in defining the correlation functions. Since the eqs. (5.14) relating χ, $\bar{\chi}$ to $f^{(2,0)}$ and $f^{(0,2)}$ respectively, now become equations of motion, we cannot start from the old observables $\Omega_{(p,q)}$, but rather from the new ones $\widetilde{\Omega}_{(p,q)}$. After performing the functional integration over the chiral superfields M_{mn}, $M_{\bar{m}\bar{n}}$ one recovers the full system of eqs. (5.14) and one can infer that (5.30) still holds (albeit with a gauge fixed action). This is important in order to make sure that we are discussing the correlation functions of the solution to the cohomology problem of q and $\bar{\mathrm{q}}$.

It is possible to develop an analysis for TYM with two fermionic charges similar to that performed in [12] in order to show that the BRS gauge fixing in superspace is equivalent to Witten's method of computing the correlation functions. Let us write the gauge-fixing action in the form

$$-\tfrac{1}{4}\,\mathrm{s}\int_{\mathcal{K}} \gamma \left\{ \mathrm{q}\,\mathrm{Tr}\, d\bar{\partial} u - \bar{\mathrm{q}}\,\mathrm{Tr}\, d^\dagger \partial \bar{u} \right\} \ . \tag{5.37}$$

Here d and $d^\dagger$ are the first components of the chiral superfields D and $D^\dagger$. The $(1,0)$ and $(0,1)$ forms

$$u = u_m \mathrm{d}z^m \ ; \qquad \bar{u} = u_{\bar{m}} \mathrm{d}z^{\bar{m}} \tag{5.38}$$

are constructed from the chiral connection superfields

$$\phi_m = u_m + \theta \pi_m \ ; \qquad \phi_{\bar{m}} = u_{\bar{m}} + \bar{\theta} \pi_{\bar{m}} \ . \tag{5.39}$$

In view of the above gauge fixing term one can start from the following path integral for the correlation functions

$$\langle \mathcal{O} \rangle = \int [\mathrm{d}\hat{\mu}][\mathrm{d}V]\, \mathcal{O}(\hat{\mu}, V)\, \exp\left\{ -\frac{1}{e^2} \mathcal{S}[\hat{\mu}, V] \right\} \delta(\nabla^m \phi_m) \delta(\nabla^{\bar{m}} \phi_{\bar{m}}) \hat{\Delta}(\phi, \bar{\phi}) \tag{5.40}$$

where $\hat{\mu}$ represents the collection of superfields ϕ_m, $\phi_{\bar{m}}$, Λ, X_{mn}, $X_{\bar{m}\bar{n}}$; $\mathcal{O}(\hat{\mu}, V)$ denotes a gauge invariant function (a product of Donaldson polynomials) and

$$\hat{\Delta}^{-1}(\phi, \bar{\phi}) = \int_{\mathcal{G}} [\mathrm{d}g]\, \delta(\nabla^m \phi^g_m) \delta(\nabla^{\bar{m}} \phi^g_{\bar{m}}) \tag{5.41}$$

is the Faddeev–Popov (super)determinant.

From now on we will express all superfields by components. We would like to consider the gauge group $\mathcal{G}$ consisting of chiral transformations with the parameters η and $\eta^\dagger$ acting on the field components as follows:

$$\begin{array}{rclrcl} u'_m &=& \mathrm{e}^{-\eta}(u_m + \nabla_m)\mathrm{e}^{\eta}\,; & u'_{\bar{m}} &=& \mathrm{e}^{\eta^\dagger}(u_{\bar{m}} + \nabla_{\bar{m}})\mathrm{e}^{-\eta^\dagger}\,; \\ \pi'_m &=& \mathrm{e}^{-\eta}\pi_m \mathrm{e}^{\eta}\,; & \pi'_{\bar{m}} &=& \mathrm{e}^{\eta^\dagger}\pi_{\bar{m}}\mathrm{e}^{-\eta^\dagger}\,; \\ \mathrm{e}^{v'} &=& \mathrm{e}^{\eta\dagger}\mathrm{e}^{v}\mathrm{e}^{\eta} & & & \end{array} \tag{5.42}$$

where v is the first component of the superfield V. The components of non-chiral gauge superfields transform according to unitary transformations generated by $h^\dagger = -h$

$$\begin{array}{rclrcl} a'_m &=& \mathrm{e}^{-h}(a_m + \nabla_m)\mathrm{e}^{h}\,; & a'_{\bar{m}} &=& \mathrm{e}^{-h}(a_{\bar{m}} + \nabla_{\bar{m}})\mathrm{e}^{h}\,; \\ \pi'_m &=& \mathrm{e}^{-h}\pi_m \mathrm{e}^{h}\,; & \pi'_{\bar{m}} &=& \mathrm{e}^{-h}\pi_{\bar{m}}\mathrm{e}^{h}\,; \\ \omega' &=& \mathrm{e}^{-h}\omega \mathrm{e}^{h}\,; & \bar{\omega}' &=& \mathrm{e}^{-h}\bar{\omega}\mathrm{e}^{h}\,; \\ \varphi' &=& \mathrm{e}^{-h}\varphi \mathrm{e}^{h}\,. & & & \end{array} \tag{5.43}$$

(As can be seen from (2.17) h depends highly non-trivially upon η, $\eta^\dagger$, and v. The components of Λ, X_{mn}, and $X_{\bar{m}\bar{n}}$ transform similarly, but they are not of interest for us here.) Finally, the matrix $\mathrm{e}^{\frac{v}{2}}$ transforms in one of the following equivalent ways

$$\mathrm{e}^{\frac{v'}{2}} = \mathrm{e}^{\eta^\dagger}\mathrm{e}^{\frac{v}{2}}\mathrm{e}^{h} = \mathrm{e}^{-h}\mathrm{e}^{\frac{v}{2}}\mathrm{e}^{\eta} \tag{5.44}$$

and serves to relate components of chiral and of gauge superfields

$$\begin{array}{rclrcl} u_m &=& \mathrm{e}^{-\frac{v}{2}}(a_m + \nabla_m)\mathrm{e}^{\frac{v}{2}}\,; & u_{\bar{m}} &=& \mathrm{e}^{\frac{v}{2}}(a_{\bar{m}} + \nabla_{\bar{m}})\mathrm{e}^{-\frac{v}{2}}\,; \\ \pi_m &=& \mathcal{D}_m\omega + \mathrm{e}^{-\frac{v}{2}}\left(\psi_m - \mathrm{D}_m\omega\right)\mathrm{e}^{\frac{v}{2}}\,; & \pi_{\bar{m}} &=& \mathcal{D}_{\bar{m}}\bar{\omega} + \mathrm{e}^{\frac{v}{2}}\left(\psi_{\bar{m}} - \mathrm{D}_{\bar{m}}\bar{\omega}\right)\mathrm{e}^{-\frac{v}{2}}\,. \end{array} \tag{5.45}$$

Let us now perform a chiral transformation on the path integral measure and on the integrand of (5.40). Everything but the δ-function is invariant under such a transformation.

If one chooses the chiral transformation such that $v' = 0$, one gets from (5.42)–(5.44) the relations

$$\begin{array}{rclrcl} \mathrm{e}^{\eta} & = & \mathrm{e}^{-\frac{v}{2}}\mathrm{e}^{h} \ ; & \mathrm{e}^{\eta^\dagger} & = & \mathrm{e}^{-h}\mathrm{e}^{-\frac{v}{2}} \ ; \\ u'_m & = & a'_m \ ; & u'_{\bar{m}} & = & a'_{\bar{m}} \ ; \\ \pi'_m & = & \psi'_m \ ; & \pi'_{\bar{m}} & = & \psi'_{\bar{m}} \ . \end{array} \tag{5.46}$$

One can show that

$$\begin{array}{rcl} V & = & 2\theta\omega' - 2\bar{\theta}\bar{\omega}' + \theta\bar{\theta}\,(\mathrm{i}\varphi' - 2\{\omega', \bar{\omega}'\}) \ ; \\ \phi_m & = & {a'}_m + \theta{\psi'}_m \ ; \qquad \phi_{\bar{m}} = {a'}_{\bar{m}} + \bar{\theta}{\psi'}_{\bar{m}} \end{array} \tag{5.47}$$

is the supersymmetry gauge in which $\mathrm{q} = \partial_\theta$ and $\bar{\mathrm{q}} = \partial_{\bar{\theta}}$ have the action given by (5.6).

Now we perform the change of field variables

$$\begin{array}{rclrcl} u_m & \rightarrow & a_m \ ; & u_{\bar{m}} & \rightarrow & a_{\bar{m}} \ ; \\ \pi_m & \rightarrow & \psi_m \ ; & \pi_{\bar{m}} & \rightarrow & \psi_{\bar{m}} \ , \end{array} \tag{5.48}$$

so that any dependence of v, ω, and $\bar{\omega}$ in $\mathcal{S}$ and $\mathcal{O}$ disappears and moreover the corresponding Jacobians are equal to one. The system of variables $\hat{\mu}$ is replaced by μ. One can easily check that

$$\begin{array}{l} \langle \mathcal{O} \rangle = \displaystyle\int [\mathrm{d}\mu] \quad \mathcal{O}(\mu) \exp\left\{-\frac{1}{e^2}\mathcal{S}[\mu]\right\} \\ \qquad\qquad \times\delta(\nabla^m a'_m)\delta(\nabla^{\bar{m}} a'_{\bar{m}})\delta(\nabla^m \psi'_m)\delta(\nabla^{\bar{m}}\psi'_{\bar{m}})\hat{\Delta}(a,\psi;\bar{a},\bar{\psi}) \end{array} \tag{5.49}$$

where the new Faddeev–Popov (super)determinant is obtained by integrating over the unitary subgroup generated by h:

$$\begin{array}{rcl} \hat{\Delta}^{-1}(a,\psi;\bar{a},\bar{\psi}) & = & \displaystyle\int [\mathrm{d}h]\, \delta(\nabla^m a'_m)\delta(\nabla^{\bar{m}} a'_{\bar{m}}) \\ & & \times\delta(\nabla^m \psi'_m)\delta(\nabla^{\bar{m}}\psi'_{\bar{m}}) \ . \end{array} \tag{5.50}$$

The above considerations lead to the following path integral for the correlation functions:

$$\langle \mathcal{O} \rangle = \int [\mathrm{d}\mu][\mathrm{d}\nu]\, \mathcal{O}(\mu)\, \exp\left\{ -\frac{1}{e^2} \mathcal{S}[\mu,\nu] \right\} \tag{5.51}$$

where

$$\begin{aligned} \mathcal{S}[\mu,\nu] \quad &= \quad \tfrac{1}{4} \int_{\mathcal{K}} \Big\{ \tfrac{1}{2} \mathrm{q}\bar{\mathrm{q}}\, \mathrm{Tr} \left(\chi\bar{\chi} - \mathrm{i}\gamma^2 \lambda f \right) \\ & \qquad - \mathrm{s}\gamma \left[\mathrm{q}\, \mathrm{Tr}\, d\, \bar{\mathrm{D}}_0 (a - a_0) - \bar{\mathrm{q}}\, \mathrm{Tr}\, d^\dagger\, \mathrm{D}_0 (\bar{a} - \bar{a}_0) \right] \Big\} \end{aligned} \tag{5.52}$$

and ν stands for the fields d, $d^\dagger$ as well as all the fields obtained from them by applying s, q, and $\bar{\mathrm{q}}$. We also introduced the background gauge field forms a_0, $\bar{a}_0$ and the corresponding covariant differentials D_0, $\bar{\mathrm{D}}_0$ upon which s, q, and $\bar{\mathrm{q}}$ act trivially.

The similarity of (5.51) with the expression used in [12] shows that the prescription to evaluate the Donaldson invariants can be derived from the standard (with gauge fixed action) path integral also for TYM with two fermionic charges. Of course, the prescription coincides with that obtained by neglecting the gauge fixing term.

A first systematic attempt to compute Donaldson invariants of smooth, oriented, compact four-manifolds has been given by Kronheimer and Mrowka [47]. They showed that the Donaldson invariants of the so-called manifols of simple type exhibit universal relations. It has been conjectured [47], [5] that all simply-connected four-manifolds with b_2^+ (b_2^+ is the number of independent self-dual forms) are of simple type. Subsequently, almost all $SU(2)$ and $SO(3)$ Donaldson invariants for Kähler four-manifolds of simple type with $b_2^+ = \dim \mathcal{H}^{(1,1)}(\mathcal{K}; \partial, \bar{\partial}) > 1$ have been calculated by Witten [4], making use of the known infrared behaviour of $N = 1$ supersymmetric gauge theories. On the other hand, precise formulas relating the Donaldson invariants to Seiberg–Witten invariants (for a review see [48]) have been conjectured in [5]. In sharp contrast to Donaldson invariants, which are defined on the moduli space of instantons, Seiberg–Witten invariants are associated to moduli spaces of abelian monopoles. The Seiberg–Witten theory is a powerful method which allows the calculation of all Donaldson invariants in case of Kähler manifolds of simple type as mentioned above. In order to make contact with the present work we point out that the manifolds of simple type can only have correlation functions of the observables $\Omega_{(1,1)}$ and $\Omega_{(2,2)}$.

A different approach based on the holomorphic Yang–Mills theory [30] has been proposed in [49] and used for computing correlation functions of the product $\Omega_{(2,0)}\Omega_{(0,2)}$.

Concerning the mathematical literature we refer to [50] where the Donaldson invariants are obtained by means of the so-called blowup formula. A first step in proving the formulas conjectured by Witten [5] has been made in [51].

Finally let us mention two papers [52], [53] where the question of computing Donaldson invariants for four-manifolds with $b_2^+ \leq 1$ is raised.

Some results of this section have been obtained in refs. [29], [30]. They concern Donaldson observables with equal ghost numbers $\Omega_{(p,p)}$. Here we included the off-diagonal Donaldson observables, thereby completing the interpretation of the fermionic charges as complex derivations on the instanton moduli space.

6 Conclusions

In the present paper we formulated TYM theory with two fermionic charges on the superspace consisting of a Kähler four-manifold and two Grassmann variables. In contrast to TYM theory with a single fermionic charge, we had to impose certain constraints in superspace. We solved the constraints and showed that the gauge transformations were replaced by local chiral transformations. Then we elucidated the structure of the Faddeev–Popov ghost sector and determined the total action.

Furthermore, we used the action for perturbatively computing the (Kähler) gravitational contribution to the dimension of the instanton moduli space. In performing the calculation we showed how the covariant point-splitting technique can be extended to Kähler manifolds.

Insisting on the specific form taken by the local conservation law on Kähler manifolds we discussed in some detail the global symmetries of the action. We showed that the associated currents representing locally these symmetries (energy-momentum tensor, fermionic and antisymmetry currents) are highest components of gauge invariant superfields. BRS symmetry does not alter this property, while the irreducibility of the multiplets is sometime lost. In any case, all their correlation functions vanish.

In addition we also determined the non-trivial observables. They are cohomology classes of both fermionic symmetry operations. Some of the classes involve additional fields, absent in TYM with a single fermionic charge. Nevertheless,

we could show that the correlation functions of all non-trivial observables can be represented as integrals of top-forms over the instanton moduli space.

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